

Test 1: Linear Systems

Math 214
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Friday March 3 2006
2:30pm-3:25pm

Name: Key

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a one hour, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and **CLEARLY** indicate your final answers to be graded from your "scratch work."

No.	Score	Maximum
1		20
2		30
3		20
4		30
BONUS		10
Total		100

1. Span, Linear Independence, Rank. 20 points.

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) (4 points.) Show that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (4 points.) Given your knowledge of $\text{rref}(A)$, what is the rank of the matrix A ? EXPLAIN YOUR ANSWER.

rank is 3 (number of non-zero rows of matrix A)

(c) (4 points.) Given your knowledge of $\text{rref}(A)$, what is the span of the columns of matrix A ? EXPLAIN YOUR ANSWER.

span of the columns = \mathbb{R}^3 since the columns must be lin ind since the rank = number of columns and no free variables

(d) (4 points.) Given your knowledge of $\text{rref}(A)$, discuss the linear independence of the columns of the matrix A . EXPLAIN YOUR ANSWER.

Columns (and rows) are linearly independent

(e) (4 points.) Given your knowledge of $\text{rref}(A)$, discuss whether the matrix A^{-1} exists. EXPLAIN YOUR ANSWER.

A^{-1} exists since $\text{rref}(A) = I$.

2. Dot product, magnitude, lengths. 30 points.

Suppose the dot product $\vec{u} \cdot \vec{v}$ is re-defined to be just the product of the lengths of the vectors \vec{u} and \vec{v} . Let's call this new dot product the **Buckmire product** and denote it

$$\vec{u} \circ \vec{v} = |\vec{u}| |\vec{v}|$$

Discuss which of the following statements are true for all vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$ under the **Buckmire product**.

(a) (6 points.) $\vec{u} \circ \vec{v} = \vec{v} \circ \vec{u}$ **TRUE**

Multiplication is commutative
This statement is true for dot product AND the Buckmire product.

(b) (6 points.) $(c\vec{u}) \circ \vec{v} = c(\vec{u} \circ \vec{v})$ **FALSE**

$$(c\vec{u}) \circ \vec{v} = |c\vec{u}| |\vec{v}| \\ = |c| |\vec{u}| |\vec{v}| \neq c |\vec{u}| |\vec{v}| \text{ if } c < 0$$

This statement is true for dot product,
FALSE for Buckmire product

(c) (6 points.) $\vec{u} \circ (\vec{v} + \vec{w}) = \vec{u} \circ \vec{v} + \vec{u} \circ \vec{w}$ **FALSE**

$$\vec{u} \circ (\vec{v} + \vec{w}) = |\vec{u}| |\vec{v} + \vec{w}|$$

$|\vec{v} + \vec{w}| \neq |\vec{v}| + |\vec{w}|$ for all \vec{v}, \vec{w}
True for dot product, not true for Buckmire product

(d) (6 points.) $\vec{u} \circ \vec{u} \geq 0$ **TRUE**

$$\vec{u} \circ \vec{u} = |\vec{u}| |\vec{u}| = |\vec{u}|^2 \geq 0$$

This is **TRUE** for Buckmire and dot product.

(e) (6 points.) $\vec{u} \circ \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$. **TRUE**

$$\text{If } \vec{u} = \vec{0} \Rightarrow |\vec{u}| = 0 \Rightarrow \vec{u} \circ \vec{u} = |\vec{u}| |\vec{u}| = 0 \cdot 0 = 0$$

$$\text{If } |\vec{u}|^2 = 0 \Rightarrow |\vec{u}| = 0 \Rightarrow \vec{u} = \vec{0}$$

TRUE for Buckmire product,

~~NOT~~ **TRUE** for dot product also

3. Matrix Operations. 20 points.

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

Recall the zero matrix \mathcal{O} and identity matrix \mathcal{I} have particular properties in matrix arithmetic which often (but not always!) correspond to the properties the number zero and the number one that you know and love.

NOTE: A is assumed to be a generic (unknown) $m \times n$ matrix for every part below.

(a) 5 points. TRUE or FALSE? "If $A^2 = \mathcal{O}$ then $A = \mathcal{O}$."

FALSE If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{O}$

There exists an $A \neq \mathcal{O}$ for which $A^2 = \mathcal{O}$

(b) 5 points. TRUE or FALSE? "If $A = \mathcal{O}$ then $A^2 = \mathcal{O}$."

FALSE $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ A^2 is not defined

If A is square, then if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
 $A^2 = A = \mathcal{O}$

(c) 5 points. TRUE or FALSE? "If $A^2 = \mathcal{I}$ then $A = \mathcal{I}$."

FALSE If $A = A^{-1}$ then $A \cdot A^{-1} = A^2 = \mathcal{I}$
 A matrix whose inverse equals itself is $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(d) 5 points. TRUE or FALSE? "If $A = \mathcal{I}$ then $A^2 = \mathcal{I}$."

TRUE $A = \mathcal{I} \Rightarrow AA = \mathcal{I} \cdot \mathcal{I} = \mathcal{I} \checkmark$

4. Parametric equations, lines, planes, subspaces. 30 points.

Consider the object described parametrically by $x = t + 1, y = 2t - 3, z = 3t$ in \mathbb{R}^3 .

(a) (10 points.) Write down a system of three linear equations which has this object as its solution.

$$\begin{aligned} x = 1 + t &\Rightarrow t = x - 1 \\ y = 2t - 3 &\Rightarrow t = \frac{y + 3}{2} \\ z = 3t &\Rightarrow t = \frac{z}{3} \end{aligned}$$

$$\begin{aligned} x - 1 &= \frac{y + 3}{2} & \frac{z}{3} &= x - 1 & \frac{z}{3} &= \frac{y + 3}{2} \\ 2x - 2 &= y + 3 & z &= 3x - 3 & 2z &= 3y + 9 \\ \boxed{2x - y} &= \boxed{5} & \boxed{-3x + z} &= \boxed{-3} & \boxed{2z - 3y} &= \boxed{9} \end{aligned}$$

(b) (10 points.) What is the dimension of this object? What is the geometric interpretation of this solution to your linear system in (a)? Write down a vector equation describing this object.

$$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 0 & 1 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 9 \end{pmatrix}$$

This is the intersection of three planes in \mathbb{R}^3 along a line not thru origin

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t + 1 \\ 2t - 3 \\ 3t \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$$

This is a 1-D object (a line).

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 5 \\ -3 & 0 & 1 & -3 \\ 0 & -3 & 2 & 9 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 0 & 5 \\ -1 & -1 & 1 & 2 \\ 0 & -3 & 2 & 9 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 2 & 9 \\ 0 & -3 & 2 & 9 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

(c) (10 points.) Is this object a subspace of \mathbb{R}^3 ? Prove your answer!

If $t=0$,
 $\vec{x} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \neq \vec{0}$

This object is NOT a subspace of \mathbb{R}^3 since the zero vector is NOT on the line.

$c\vec{x} = c \left(t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \right)$
 Not on the line

It is also NOT closed under scalar ~~add~~ multiplication or vector addition.

$\vec{x} + \vec{y} = s \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} = (s+t) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix}$ is in same direction but not on line!

BONUS QUESTION. Linear Independence, Dependence, Inverse. (10 points.)

If possible, write down five different 3×3 matrices each one which has one of the following properties:

- (i) MATRIX A: The rows are linearly independent but the columns are linearly dependent.
- (ii) MATRIX B: The rows are linearly dependent but the columns are linearly independent.
- (iii) MATRIX C: The rows are linearly independent but the columns are linearly dependent.
- (iv) MATRIX D: The rows are linearly dependent but the columns are linearly independent.
- (v) MATRIX E: The transpose of the matrix equals the inverse of the matrix.

EXPLAIN YOUR ANSWER THOROUGHLY. EXTRA CREDIT POINTS ARE HARD TO GET.

(i) Use example from Question 1.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$\text{rref}(A) = I$ so
columns & rows
are lin indep

(ii) $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

rows are
lin ind

(iii) $B = \begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{2} & \cancel{2} & \cancel{2} \\ \cancel{3} & \cancel{3} & \cancel{3} \end{pmatrix} C^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix}$

(iv) $D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

cols & rows are all
the same so you have
lin dep rows and cols

(v) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = E$ (or any permutation matrix)