FINAL EXAM: Linear Systems

Tuesday May 6, 2008: 1:00-4:00pm Math 214

Name: Key

Directions: There are three sections to this exam. The first section consists of multiple choice questions, similiar to the Clicker Questions we did in class. The second section consists of Definitions and Proofs. The third section is more related to specific concepts and calculations.

This exam is a closed-notes, closed-book, test. No calculators.

In **Part I** You must clearly write the letter of your chosen answer on the line provided.

In **Part II** You must provide a definition which is clear enough so that a student at the beginning of the class would be able to understand what you mean. This may require defining secondary terms that you use in your own definition.

In **Part III** You must include ALL relevant work to support your answers. CLEARLY indicate your final answer from your "scratch work.'

Pledge: I, ______________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
PART 1 (Multiple Choice)		**40**
PART II (Proofs and Definitions)		**60**
II.1		20
II.2		20
II.3		20
PART III (Concepts and Calculations)		**100**
III.1		20
III.2		20
III.3		20
III.4		20
III.5		20
BONUS		10
TOTAL		**200**

FINAL EXAM: Linear Systems (PART I)

1. How many non-zero vectors in $\mathbb{R}^3$ must one have in a set of vectors in order to be confident the given set of vectors is linearly independent?

(a) One

(b) Two

(c) Three

(d) As Many As We Want ANS: A .

2. What can we say about two non-zero vectors whose dot product is positive?

(a) The vectors are orthogonal to each other.

(b) The angle between the two vectors is greater than 90^o.

(c) The angle between the two vectors is less than 90^o.

(d) There's not enough information to say anthing about the vectors. ANS: C .

3. A linear systems of 5 equations in 3 unknowns can have

(a) No Solution.

(b) One Solution.

(c) An Infinite Number of Solutions.

(d) Any Of The Above Could Be True. ANS: D

4. In order for the linear system $A\vec{x} = \vec{b}$ to have a solution, the vector $\vec{b}$ must be in the

(a) row space of A, row(A)

(b) column space of A, col(A)

(c) null space of A, null(A)

(d) left null space of A, null(A^T) ANS: B

5. The vector(s) $\vec{x}$ which satisfy the homogeneous equation $A\vec{x} = \vec{0}$ must be in the

(a) row space of A, row(A)

(b) column space of A, col(A)

(c) null space of A, null(A)

(d) left null space of A, null(A^T) ANS: C

FINAL EXAM: Linear Systems (PART I)

6. If $\vec{b} = (3, -1)$ and $\vec{y} = (2, 1)$, then the projection of $\vec{b}$ onto $\vec{y}$, $\text{proj}_{\vec{y}}(\vec{b})$ is

(a) $(2, 1)$

(b) $(3/2, -1/2)$

(c) $(10, 5)$

(d) $(1/10, 3/10)$

$\left(\frac{\vec{b}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\right)\vec{y} = \frac{\binom{2}{1}\cdot\binom{3}{-1}}{\binom{2}{1}\binom{2}{1}}\binom{2}{1} = \frac{6-1}{4+1}\binom{2}{1} = \frac{5}{5}\binom{2}{1}$

ANS: A.

7. A general equation of the plane represented by the span$\{(1, 0, 1), (-1, 1, 0)\}$ is

(a) $x + y + z = 0$.

(b) $x + y - z = 0$.

(c) $\vec{x} = s(1, 0, 1) + t(-1, 1, 0)$.

(d) $x = s, y = t, z = s + t$.

ANS: B.

8. If one multiplies one row of a square $n \times n$ matrix A by a factor of 2 to produce a matrix B then $\det(B)$ will equal

(a) $\det(A)$

(b) $2\det(A)$

(c) $2^n\det(A)$

(d) Impossible to determine from the information given.

ANS: B

9. The eigenvalues of the matrix $A = \begin{bmatrix} -1 & -3 \\ 2 & -6 \end{bmatrix}$ are

$(-1-\lambda)(-6-\lambda) - (-6) =$
$\lambda^2 + 7\lambda + 12 = 0$
$(\lambda+3)(\lambda+4) = 0$

(a) -3 and 4.

(b) 3 and -4.

(c) -3 and -4.

(d) 3 and 4.

ANS: C

10. If $A^2 = \mathcal{I}$ where $\mathcal{I}$ is the identity matrix, then A must equal

(a) $\mathcal{I}$

(b) A^{-1}.

(c) A^T.

(d) Impossible to determine from the information given.

ANS: B

FINAL EXAM: Linear Systems (PART I)

11. Which of the following vector pairs can be the eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(a) $\{(-1,1),(1,1)\}$.

must have eigenvalue 0

(b) $\{(0,1),(1,0)\}$.

(c) $\{(2,3),(-3,2)\}$.

(d) $\{(2,1),(-1,2)\}$. ANS: A

12. Which of the following is **NOT** a basis for the row space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$?

(a) $\{(1,0),(0,1)\}$

(b) $\{(1,2),(3,4)\}$

(c) $\{(1,2),(2,4)\}$

(d) $\{1,2),(1,4)\}$ ANS: C.

13. The difference between a span and a basis for the same subspace in $\mathbb{R}^n$ is

(a) a basis will always contain more vectors than the span

(b) a basis will always contain linearly independent vectors

(c) a basis will contain the zero vector, a span will not

(d) There is no difference between a span and a basis. ANS: B

14. If a $n \times n$ matrix A is diagonalizable, then

(a) A is invertible.

(b) A is NOT invertible.

(c) A must possess n different eigenvalues.

(d) None of the above statements is true. ANS: D

15. If A is an invertible matrix and $AB = \mathcal{O}$ where $\mathcal{O}$ is the zero matrix, then

(a) A must equal the zero matrix.

(b) B must equal the zero matrix.

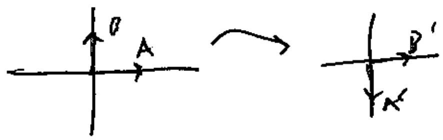
(c) Either A or B must equal the zero matrix.

(d) None of the above statements is true. ANS: B

FINAL EXAM: Linear Systems (PART I)

16. The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ to rotate a vector $\vec{x} = (x, y)$ 90^o clock-wise about the origin is represented by the standard matrix

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$ $\quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$



$A = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

ANS: D.

17. The least squares solution to the linear system $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a) does not exist

(b) is $x = 0, \quad y = -1/5$

(c) is $x = -1/5, \quad y = 0$

(d) is $x = -1/5, \quad y = -1/5$.

$A^T\vec{b} = \begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{matrix}-1\\-2\end{matrix}$

$A^TA = \begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix} = \left(\begin{array}{cc|c}5 & 10 & -1\\10 & 5 & -2\end{array}\right) \to \left(\begin{array}{cc|c}1 & 2 & -1/5\\2 & 1 & -2/5\end{array}\right)$

ANS: C.

18. If the rank of a 4x4 matrix is equal to 3, then

(a) the matrix is invertible.

(b) the dimension of the null space is 4.

(c) the dimension of the null space is 3.

(d) the dimension of the row space is 3.

ANS: D

19. Every real symmetric matrix is

(a) an invertible matrix.

(b) a diagonalizable matrix.

(c) an orthogonal matrix.

(d) an elementary matrix.

ANS: B

20. The solution $\vec{x}$ of the linear system $A\vec{x} = \vec{b}$ is able to written as the sum of two vectors, one each from the

(a) column space and row space.

(b) column space and null space.

(c) row space and null space.

(d) column space and left null space.

ANS: C

FINAL EXAM: Linear Systems (PART II)

II.1 *[20 points total.]*
a. (*12 points*). Give the definitions of the column space, row space, and null space of a $m \times n$ matrix A .

The columnspace is the set of all possible linear combinations of the columns of A, it is a subspace of $\mathbb{R}^m$.

A Linear combination of vectors is the sum of scalar multiples of those vectors.

The rowspace is the set of all possible linear combinations of the rows of A, it is a subspace of $\mathbb{R}^n$.

The null space is the set of all solutions to $A\vec{x} = \vec{0}$, it is a subspace of $\mathbb{R}^n$.

b. (*8 points*). Given a $m \times n$ matrix A and a vector $\vec{b}$, if the equation $A\vec{x} = \vec{b}$ has a solution, then is $\vec{b}$ necessarily in col(A), row(A), null(A), or none of the above? Explain your reasoning.

If $A\vec{x} = \vec{b}$ has a solution then there exists an $\vec{x} \in \mathbb{R}^n$ such that $\vec{b} = A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} \text{col}_1 A \\ \\ \end{pmatrix} + x_2 \begin{pmatrix} \text{col}_2 A \\ \\ \end{pmatrix} + \dots + x_n \begin{pmatrix} \text{col}_n A \\ \\ \end{pmatrix}$

Since $\vec{b}$ can be written as a linear combination of the columns of A, it is a member of col(A).

$\vec{b} \notin$ row(A) because row(A) $\subset \mathbb{R}^n$ not $\mathbb{R}^m$

$\vec{b} \notin$ null(A) because null(A) $\subset \mathbb{R}^n$ not $\mathbb{R}^m$

Even if A was square (i.e. $n \times n$) ~~If~~ since $\vec{b} \in$ col(A) it can not be in any other vector space simultaneously, unless $b = \vec{0}$.

FINAL EXAM: Linear Systems (PART II)

II.2 *[20 points total.]*

a. (*4 points*). Give the definition of the term "An eigenvector $\vec{v}$ of a matrix A."

"An eigenvector $\vec{v}$ of a matrix A" is a vector which satisfies the equation $A\vec{v} = \lambda\vec{v}$, in other words $A\vec{v}$ is a scalar multiple of $\vec{v}$.

b. (*4 points*). Give the definition of the term "The inverse of the matrix A."

"The inverse of the matrix A (if it exists) is a matrix M (also square) which satisfies the equation

$$AM = MA = I$$

where I is the identity matrix a diagonal matrix with all ones along the diagonal.

c. (*4 points*). Give the definition of the term "An eigenvalue of the matrix A."

"An eigenvalue of the matrix A" is a solution of the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$ where $\det(A-\lambda I)$ is the determinant of $A - \lambda I$. The determinant function is a function which takes a nxn matrix as input and outputs a real number. In $A\vec{v} = \lambda\vec{v}$, λ is the eigenvalue.

d. (*8 points*). Prove that if λ is an eigenvalue of matrix A and A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of the inverse of A.

$$A\vec{x} = \lambda\vec{x}$$

A^{-1} exists (so $\lambda \neq 0$)

$$A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x})$$

$$\vec{x} = \lambda(A^{-1}\vec{x})$$

$$\frac{1}{\lambda}\vec{x} = \cancel{\lambda} A^{-1}\vec{x}$$

$\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

FINAL EXAM: Linear Systems (PART II)

II.3 *[20 points total.]*

a. *(4 points).* Give the definition of the term "a subspace of $\mathbb{R}^n$."

"A subspace of $\mathbb{R}^n$" is a subset of $\mathbb{R}^n$ which

(i) contains $\vec{0}$ (i.e. $\vec{0} \in V$)

(ii) is closed under scalar multiplication
i.e. $\forall \vec{v} \in V \Rightarrow (c\vec{v} \in V)$

(iii) is closed under vector addition
i.e. $\forall \vec{v}, \vec{w} \in V \Rightarrow (\vec{v} + \vec{w}) \in V$

b. *(4 points).* Give the definition of the term "span."

The span of a ~~set~~ collection of vectors is the set of all possible linear combinations of those vectors. This set of vectors is said to "span" the given space.

c. *(4 points).* Give the definition of the term "basis."

A basis for a subspace is a set of vectors ~~which~~ whose span equals the subspace and whose vectors are linearly independent.

Linearly independent means that a set of vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ has only the trivial solution to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \ldots c_n\vec{v}_n = \vec{0}$,
i.e. $c_1 = c_2 = c_3 = \ldots = c_n = 0$.

d. *(8 points).* Prove that the span of a collection of k vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Let $S = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k)$

Any $\vec{s} \in S$ can be written $\vec{s} = \sum_{i=1}^{k} c_i \vec{v}_i$ where c_i is ALL REAL NUMBERS

(i) Since c_i could all equal 0, $\vec{s} = 0 \in S$

~~(ii) Given $\vec{s} = \sum_{i=1}^{k} c_i \vec{v}_i$~~

$d\vec{s} = d\sum_{i=1}^{k} c_i \vec{v}_i$

$= \sum_{i=1}^{k} d c_i \vec{v}_i = \sum_{i=1}^{k} (dc_i)\vec{v}_i$

Since dc_i can be any real # $d\vec{s} \in S$

S is closed under scalar multiplication

(iii) Given $\vec{s} = \sum_{i=1}^{k} c_i \vec{v}_i$

and $\vec{t} = \sum_{i=1}^{k} b_i \vec{v}_i$

$\vec{s} + \vec{t} = \sum_{i=1}^{k} c_i \vec{v} + \sum_{i=1}^{k} b_i \vec{v} = \sum_{i=1}^{k} (c_i + b_i)\vec{v}_i$

$c_i + b_i$ can be any real #, so $\vec{s} + \vec{t} \in S$

S is closed under vector addition

FINAL EXAM: Linear Systems (PART III)

III.1 *[20 points total.]*
Consider the 4×4 matrix

$$M = \begin{bmatrix} 0 & a & b & d \\ -a & a & c & e \\ -b & -c & 0 & 0 \\ -d & -e & 0 & 0 \end{bmatrix}$$

(a) (*8 points.*) Think of $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as a block matrix consisting of four 2×2 matrices A, B, C and D. Find the determinant of each of the matrices A, B, C and D.

$A = \begin{pmatrix} 0 & a \\ -a & a \end{pmatrix}$ $\quad \det(A) = 0 \cdot a - (-a)(a) = a^2$

$B = \begin{pmatrix} b & d \\ c & e \end{pmatrix}$ $\quad \det(B) = b \cdot e - d \cdot c = be - dc$

$C = \begin{pmatrix} -b & -c \\ -d & -e \end{pmatrix}$ $\quad \det(C) = (-b)(-e) - (-d)(-c) = be - dc$

$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \det(D) = 0.$

(b) (*8 points.*) Show that the determinant of $M = (be - dc)^2$ using the Laplace Expansion Formula (pick the row or column you expand about carefully!)

Using the 3rd column!

$$\det(M) = (-1)^{3+1} b \begin{vmatrix} -a & a & e \\ -b & -c & 0 \\ -d & -e & 0 \end{vmatrix} + (-1)^{3+2} c \begin{vmatrix} 0 & a & d \\ -b & -c & 0 \\ -d & -e & 0 \end{vmatrix}$$

$$= b \begin{vmatrix} -a & a & e \\ -b & -c & 0 \\ -d & -e & 0 \end{vmatrix} - c \begin{vmatrix} 0 & a & d \\ -b & -c & 0 \\ -d & -e & 0 \end{vmatrix} = b(-1)^{1+3} e \begin{vmatrix} -b & -c \\ -d & -e \end{vmatrix} - c \cdot d (-1)^{1+3} \begin{vmatrix} -b & -c \\ -d & -e \end{vmatrix}$$

$$= +be\,(be - dc) - dc\,(be - dc) = (be - dc)^2$$

(c) (*4 points.*) Write down a relationship between the determinant of the block matrix M, i.e. $\det(M)$ and the determinants of the blocks, i.e. $\det(A)$, $\det(B)$, $\det(C)$, and $\det(D)$. [HINT: this may not be a relationship that you expect!]

$$\det(M) = (be - dc)^2 = \det(B)\det(C) - \det(A)\det(D)$$

$$= (be - dc)(be - dc) - a^2 \cdot 0$$

$$= (be - dc)^2$$

Block Matrices are weird! $M = \begin{matrix} A & B \\ C & D \end{matrix}$

III.2 *[20 points]* Multiple Matching. Write the letter (or letters) of ALL statements which are always true when the numbered statement given is true. (Note: Anywhere from none to all of the statements may match for each one. I've also given you lots of space to work, but you do not necessarily need to show any work on this problem – I will just be checking your answers in the blanks provided.)

Assume A is an $m \times n$ matrix with rank r.

A B D E F G H (1) $m = n = r$. $\quad \mathrm{rref}(A) = I$

~~B~~ C H (2) $m = n > r$. $\quad \begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ 0 & 0 & 0 & 0 \end{pmatrix}$

B ~~C~~ D F G H (3) $m > n = r$. $\quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

~~B~~ C E H (4) $n > m = r$. $\quad \begin{pmatrix} 1 & & & & x & x \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \end{pmatrix}$

C H (5) $n > m > r$. $\quad \begin{pmatrix} 1 & & & x & x \\ & 1 & & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(A) For every $\vec{b}$ in $\mathbb{R}^m$ there exists at least one solution to $A\vec{x} = \vec{b}$.

(B) Assuming $\vec{b}$ is in $\mathtt{col}(A)$, $A\vec{x} = \vec{b}$ has a unique solution.

(C) Assuming $\vec{b}$ is in $\mathtt{col}(A)$, $A\vec{x} = \vec{b}$ has an infinite number of solutions.

(D) The columns of A are linearly independent.

(E) $\mathtt{col}(A) = \mathbb{R}^m$. $\quad r = m$

(F) $\mathtt{null}(A) = \{\vec{0}\}$. $\quad r = n$

(G) $\mathtt{row}(A) = \mathbb{R}^n$. $\quad r = n$

(H) There exists at least one solution to $A\vec{x} = \vec{0}$.

FINAL EXAM: Linear Systems (PART III)

III.3 *[20 points total.]*
The problems on this page and the next all refer to the following matrix and its row reduced echelon form:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & -1 \\ 3 & 2 & -1 \end{bmatrix} \qquad \texttt{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) *2 points.* Find a basis for the row space of this matrix, $\texttt{row}(A)$.

$$\text{row}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

(b) *3 points.* The row space is a 2-dimensional vector subspace of $\mathbb{R}^{3}$. We will call the row space W.

W can be represented geometrically as a PLANE IN $\mathbb{R}^3$. (Make sure you filled in all three blanks; I'm looking for a number in the first two and the name of a geometric object in the last.)

(c) *2 points.* Write an equation for the geometric figure in (b).

$$\vec{x} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \quad s,t \in \mathbb{R}$$

(d) *3 points.* The orthogonal complement of W, denoted $W^{\perp}$, is a 1-dimensional vector subspace of $\mathbb{R}^{3}$.

It can be represented geometrically as a LINE.

(e) *2 points.* Write an equation for the geometric figure in (d).

The nullspace is orthogonal to the rowspace

$$\vec{x} = s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}$$

(f) *2 points.* What is a basis for $W^{\perp}$?

$$W^{\perp} = \text{span}\left\{\begin{pmatrix}-1\\2\\1\end{pmatrix}\right\}$$

(g) *4 points.* How could you have answered part (f) directly from examining the matrices A or rref(A) rather than working through the above steps? Explain fully.

Since $W^{\perp}$ is orthogonal to W and W is the rowspace, $W^{\perp}$ is the nullspace, so then you can use the "magic" way on rref(A) to find a basis for null(A).

(h) *2 points.* Write the vector $\vec{v} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$ as the sum of two vectors, one in W and one in $W^{\perp}$.

$$\vec{w}^{\perp} = \text{proj}_{W^{\perp}}(\vec{v}) = \frac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}}{\begin{pmatrix}-1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}}\begin{pmatrix}-1\\2\\1\end{pmatrix} = \frac{-1+4-1}{1+4+1}\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}-1\\2\\1\end{pmatrix} = \vec{w}^{\perp}$$

$$\vec{w} + \vec{w}^{\perp} = \vec{v}$$

$$\vec{w} = \vec{v} - \vec{w}^{\perp}$$

$$= \begin{pmatrix}1\\2\\-1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}4/3\\4/3\\-4/3\end{pmatrix} = \frac{4}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Check $\frac{4}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}-1\\2\\1\end{pmatrix} = -1+2-1 = 0$ ✓

$\vec{w}$ and $\vec{w}^{\perp}$ are orthogonal.

FINAL EXAM: Linear Systems (PART III)

III.4 *[20 points]* Mark each of the following "TRUE" or "FALSE". If I can't tell whether you wrote "T" or "F", you will not get credit for your answer.

(a) ____F____ There exists an orthogonal matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

(b) ____F____ $\mathtt{det}(A + I) = \mathtt{det}(A) + 1$

(c) ____T____ If you have n orthogonal vectors in $\mathbb{R}^n$, the vectors must form a basis for $\mathbb{R}^n$.

(d) ____T____ If Q is an orthogonal matrix, then $\mathtt{nullity}(Q) = 0$.

(e) ____T____ If $A\vec{x} = 3\vec{x}$ for some non-zero vector $\vec{x}$, then $\lambda = 3$ must be an eigenvalue of the matrix A.

(f) ____F____ $3x - y + 2z = 4$ is the equation of a line in $\mathbb{R}^3$.

(g) ____F____ In $\mathbb{R}^3$, the span of $\vec{v}_1$ and $\vec{v}_2$, $\mathtt{span}(\vec{v}_1, \vec{v}_2)$, is always a plane through the origin.

(h) ____T____ Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by a matrix transformation using an $m \times n$ matrix.

(i) ____T____ If a matrix has more rows than columns, then its **rows** must be **linearly dependent**.

(j) ____F____ If a matrix has more rows than columns, then its **columns** must be **linearly independent**.

FINAL EXAM: Linear Systems (PART III)

III.5 *[20 points]* Below are four distinct factorizations of the matrix A. In answering the following questions (on this page and the next), you may only refer to information that is provided below in these factorizations and reason from this information. You may not multiply the matrices below and actually calculate A or do any explicit calculations from A itself. Answers that rely on such new calculations will not be accepted.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 0 & -3 & -6 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = Q D Q^T$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

(a) *4 points.* What are the eigenvalues of A? For each eigenvalue, what is a basis for its associated eigenspace?

$A = S \Lambda S^{-1} = Q D Q^T$

$\lambda = -3, 3, 3$

$E_{-3} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$E_3 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) *4 points.* Is A symmetric? Why or why not?

Yes, A is symmetric. $A = QDQ^T$

Since A can be orthogonally diagonalized, A is symmetric.

(c) *4 points.* Is A invertible? Why or why not?

$$\det(A) = \prod_{i=1}^{n} \lambda_i = 3 \cdot 3 \cdot -3 = -27 \neq 0$$

Since the $\det(A) \neq 0 \iff A^{-1}$ exists (A is invertible)

(d) *4 points.* Which of the four factorizations given would be the most efficient to use to solve $A\vec{x} = \vec{b}$? Explain fully why this factorization is the most efficient by actually describing the solution process – you don't actually have to do the calculations though (since you don't have $\vec{b}$).

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & -2 \\ 0 & -7 & -6 \\ 0 & 0 & 9 \end{pmatrix} = LU$$

$A\vec{x} = \vec{b} \Rightarrow LU\vec{x} = \vec{b}$

Let $U\vec{x} = \vec{y}$

then $L\vec{y} = \vec{b}$

Very easy to solve $L\vec{y} = \vec{b}$ for $\vec{y}$ since lower triangular then also easy to solve $U\vec{x} = \vec{y}$ to get $\vec{x}$ (since upper triangular)

(e) *4 points.* Which of the four factorizations given would be the most efficient to use to find A^{10}? Explain fully why this factorization is the most efficient by actually describing the solution process – you don't actually have to do the calculations though.

$A = QDQ^T$

$A^{10} = QD^{10}Q^T$ (since $QQ^T = I$)

This is the easiest factorization to exponentiate the matrix.

FINAL EXAM: Linear Systems (BONUS)

BONUS *[10 points total.]*
Prove that IF $\vec{x}$ and $\vec{y}$ are eigenvectors of a square, real symmetric matrix A corresponding to two different eigenvalues λ_x and λ_y respectively, THEN $\vec{x} \cdot \vec{y} = 0$.

OR

Write down a matrix A whose complete solution to $A\vec{x} = \begin{bmatrix} 3 \\ -5 \\ 5 \end{bmatrix}$ has the form

$\vec{x} = \begin{bmatrix} 3 \\ -5 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Explain your reasoning and show all your work.

The complete solution to a linear system consists of the sum of the solutions to the homogeneous and non-homogeneous linear systems.

$A\vec{x} = \lambda_x \vec{x}$ $\qquad \vec{x}\cdot\vec{y} = \vec{x}^T\vec{y}$

$A\vec{y} = \lambda_y \vec{y}$

Take transpose of $A\vec{x} = \lambda\vec{x}$ $\quad (A\vec{x})^T = (\lambda\vec{x})^T \Rightarrow \vec{x}^T A^T = \lambda_x \vec{x}^T$

(since $A = A^T$) $\vec{x}^T A = \lambda_x \vec{x}^T$

$A\vec{y} = \lambda\vec{y}$ Multiply from left by x^T $(1\times n)$

Multiply $x^T A = \lambda_x \vec{x}^T$ from right by $\vec{y}$ $(n\times 1)$ $\quad \vec{x}^T A y = \vec{x}^T(\lambda_y \vec{y}) = \vec{x}^T\vec{y}\lambda_y$

$\vec{x}^T A \vec{y} = \lambda_x \vec{x}^T \vec{y}$

$\vec{x}^T A \vec{y} = \lambda_y \vec{x}^T \vec{y}$

Subtract! $\quad 0 = (\lambda_x - \lambda_y)\vec{x}^T\vec{y}$ $\qquad$ Since $\lambda_x - \lambda_y \neq 0$, $x^T y = 0$ so $\vec{x}\cdot\vec{y} = 0$!

We know rref(A):
$\vec{x} \in \mathbb{R}^4$ or 4×1 vector
A must be a 3×4.
(since $\vec{b}$ is a 3×1)

$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$

rank(A) = 2
dim null(A) = 4 − 2 = 2

Let $A = \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & +1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$ ✓

$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓ $\qquad \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓