Class 29: Wednesday April 11

TITLE Projection Matrices and Orthogonal Diagonalization
CURRENT READING Poole 5.4

Summary
How to find the projection matrix for projection onto a subspace.

Homework Assignment
HW #27: Poole, Section 5.4: 1, 6, 7, 8, 9, 11, 12, 13, 14, 22, 23. EXTRA CREDIT 25.

1. Recalling Projection of a Vector onto Another Vector
For any vectors \( \vec{u} \) and \( \vec{v} \) where \( \vec{u} \neq 0 \) then **the projection of \( \vec{v} \) onto \( \vec{u} \)** is the vector \( \text{proj}_{\vec{u}}(\vec{v}) \) defined by:

\[
\text{proj}_{\vec{u}}(\vec{v}) = \left( \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u} = \left( \frac{\vec{u}^T \vec{v}}{\vec{u}^T \vec{u}} \right) \vec{u}
\]

(Since \( \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = \vec{b}^T \vec{a} \))

2. Projecting a Vector Onto A Subspace
What are the projections of \( \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \) onto

(a) the \( z \)-axis, i.e. span \( \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \)?

(b) the \( xy \)-plane, i.e. span \( \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \)?

We can find \( P \), a projection matrix, which finds the vector \( \vec{p} = P\vec{b} \) which is the projection of \( \vec{b} \) onto a vector space (e.g. the \( z \) axis, the \( xy \)-plane) spanned by a given basis \( \mathcal{W} \) where \( \vec{p} = \text{proj}_{\mathcal{W}}(\vec{b}) \).

Consider \( P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and \( P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). Do these matrices do what we want them to?

Note that the \( xy \)-plane is the column space of a matrix \( A \), \( \text{col}(A) \) where \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \), a matrix whose columns are a basis for the subspace consisting of the \( xy \)-plane.

Also note that the \( z \)-axis is the column space of the matrix \( A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \)

In general, we are trying to find the projection of a vector \( \vec{b} \) onto the column space of any \( m \times n \) matrix (or the span of a given basis for a vector space).
3. Computing The Projection Matrix

Let’s look more closely at that projection formula again:

\[
\text{proj}_u(\vec{v}) = \left( \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u}
\]

\[
= \left( \frac{\vec{u}^T \vec{v}}{\vec{u}^T \vec{u}} \right) \vec{u}
\]

\[
= \left( \frac{(\vec{u}^T \vec{v}) \vec{u}}{\vec{u}^T \vec{u}} \right)
\]

\[
= \left( \frac{\vec{u}(\vec{u}^T \vec{v})}{\vec{u}^T \vec{u}} \right)
\]

\[
= \left( \frac{\vec{u}^T \vec{v}}{\vec{u}^T \vec{u}} \right)
\]

\[
= \left( \frac{\vec{u}(\vec{u}^T \vec{v})}{\vec{u}^T \vec{u}} \right)
\]

\[
= \left( \frac{\vec{u}^T \vec{v}}{\vec{u}^T \vec{u}} \right)
\]

\[
= \left( \frac{\vec{u} \vec{u}^T \vec{v}}{\vec{u}^T \vec{u}} \right)
\]

\[
= \left( \frac{\vec{v}}{\vec{u}^T \vec{u}} \right)
\]

So the projection matrix \( P \) for projecting a vector \( \vec{v} \) onto a vector \( \vec{u} \) is given by \( P = \frac{\vec{u} \vec{u}^T}{\vec{u}^T \vec{u}} \).

**Exercise**

Use the above formula to find the projection matrix for the projection of the vector \( \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \) onto the \( z \)-axis.

**EXAMPLE**

Let’s change the formula to find the projection matrix onto the subspace spanned by span \( \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \), i.e. the \( xy \)-plane.

So the projection matrix \( P \) for projecting a vector \( \vec{v} \) onto a subspace equal to the \( \text{col}(A) \) is given by \( P = A(A^T A)^{-1} A^T \).
Exercise
Find the Projection matrix $P$ which computes the projection of a vector $\vec{v}$ onto the subspace $W$ where $W$ is the plane $x - y + 2z = 0$. Use your answer to obtain the projection of $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ onto $W$. After doing that, we can also find the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $W$ without doing much work. Why?

4. Orthogonal Diagonalization

**Definition**
A square matrix $A$ is **orthogonally diagonalizable** if there exists an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $Q^T A Q = D$ (or $A = QDQ^T$.)

**Example**
Let’s show the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ is orthogonally diagonalizable.

**Theorem 5.20**
A real square matrix is symmetric IF and ONLY IF it is orthogonally diagonalizable. This result is known as **The Spectral Theorem**.
**DEFINITION**

The **spectrum** of a matrix is the set of all eigenvalues of the matrix. The **spectral decomposition** of a matrix $A$ is the expression $A = \sum_{i=1}^{n} \lambda_i \vec{q}_i \vec{q}_i^T$ where $\vec{q}_1, \vec{q}_2, \vec{q}_3, \ldots, \vec{q}_n$ are the columns of the orthogonal matrix $Q$.

$$A = QDQ^T$$

$$= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \ldots & \vec{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \\ \vdots \\ \vec{q}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{q}_1 & \lambda_2 \vec{q}_2 & \lambda_3 \vec{q}_3 & \ldots & \lambda_n \vec{q}_n \end{bmatrix}$$

$$= \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \ldots + \lambda_n \vec{q}_n \vec{q}_n^T = \sum_{i=1}^{n} \lambda_i \vec{q}_i \vec{q}_i^T$$

Does this expression look familiar? Each of the expressions $\vec{q}_i \vec{q}_i^T$ are rank 1 matrices which represent the projection matrix for projecting onto the 1-dimension space spanned by each $\vec{q}_i$. (Recall $\vec{q}_i^T \vec{q}_i = 1$). The above expression is thus sometimes known as the **projection form of the Spectral Theorem**.

**Exercise**

Let’s find the spectral decomposition of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$. 