Class 27: Friday April 6

TITLE Orthogonal Complements and Orthogonal Projections

CURRENT READING Poole 5.1

Summary
We will learn about an incredibly important feature of vectors and orthogonal vector spaces.

Homework Assignment
HW#25 Poole, Section 5.2: 2, 3, 4, 5, 6, 7, 12, 15, 16, 17, 19, 20, 21. EXTRA CREDIT 29.

DEFINITION
Two subspaces \( V \) and \( W \) are said to be orthogonal if every vector \( \vec{v} \in V \) is perpendicular to every vector \( \vec{w} \in W \). The orthogonal complement of a subspace \( V \) contains EVERY vector that is perpendicular to (vectors in) \( V \). This space is denoted \( V^\perp \). In other words, \( \vec{v} \cdot \vec{w} = 0 \) or \( \vec{v}^T \vec{w} = 0 \) for every \( \vec{v} \) in \( V \) and \( \vec{w} \) in \( W \).

\[
W^\perp = \{ \vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \text{ in } W \}
\]

Example 1. Q: In \( \mathbb{R}^3 \), let \( V \) = the z-axis. What is \( V^\perp \)? A: 

Q: In \( \mathbb{R}^3 \), what is the orthogonal complement of the xy-plane? A: 

Q: In \( \mathbb{R}^3 \), are the xy-plane and the yz-plane orthogonal complements of each other? A: No, there are vectors in one plane that are not perpendicular to vectors in the other plane. (Can you find one of each?)

Q: In \( \mathbb{R}^4 \) (with axes \( x_1, x_2, x_3, x_4 \)), what is the orthogonal complement of the \( x_1x_2 \)-plane? A: 

We can summarize some of the properties of orthogonal complements.

**Theorem 5.9**
Let \( W \) be a subspace of \( \mathbb{R}^n \).

[a.] \( W^\perp \) is a subspace of \( \mathbb{R}^n \)

[b.] \( (W^\perp)^\perp = W \)

[c.] \( (W^\perp) \cap W = \vec{0} \)

[d.] If \( W = \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n) \) then \( \vec{v} \) is in \( W^\perp \) only if \( \vec{v} \cdot \vec{w}_i = 0 \) for every \( \vec{w}_i \) in \( W \) for \( i = 1 \ldots n \)

These features can be described using the associated subspaces of an \( m \times n \) matrix \( A \).

**Theorem 5.10**
Let \( A \) be an \( m \times n \) matrix. Then the orthogonal complement of the row space of \( A \) is the null space of \( A \). The orthogonal complement of the column space of \( A \) is the null space of \( A^T \) (sometimes called the left null space). Mathematically, this can be written:

\[
(\text{row}(A))^\perp = \text{null}(A) \text{ and } (\text{col}(A))^\perp = \text{null}(A^T)
\]

These four subspaces are called the fundamental subspaces of the matrix \( A \).
EXAMPLE

Let’s find bases for the four fundamental subspaces of the matrix \( A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \).

Suppose we know that \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) and \( \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Write down the dimensions of each fundamental subspace and describe the subspace-orthogonal complement pairs.

DEFINITION

Let \( \mathcal{W} \) be a subspace of \( \mathbb{R}^n \) and let \( \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n\} \) be an orthogonal basis for \( \mathcal{W} \). For any vector \( \vec{v} \) in \( \mathbb{R}^n \), the orthogonal project of \( \vec{v} \) onto \( \mathcal{W} \) is defined as

\[
\text{proj}_{\mathcal{W}}(\vec{v}) = \sum_{j=1}^{n} \text{proj}_{w_j}(\vec{v}) = \sum_{j=1}^{n} \frac{\vec{v} \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \vec{w}_j
\]

The component of \( \vec{v} \) orthogonal to \( \mathcal{W} \) is the vector \( \text{perp}_{\mathcal{W}}(\vec{v}) = \vec{v} - \text{proj}_{\mathcal{W}}(\vec{v}) \)

NOTE: this implies that \( \vec{v} = \text{perp}_{\mathcal{W}}(\vec{v}) + \text{proj}_{\mathcal{W}}(\vec{v}) \) (Draw a picture in \( \mathbb{R}^2 \)!)

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**Theorem 5.11**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $\vec{v}$ be ANY vector in $\mathbb{R}^n$. THEN there exist unique vectors $\vec{w}$ in $\mathcal{W}$ and $\vec{w}^\perp$ in $\mathcal{W}^\perp$ such that $\vec{v} = \vec{w} + \vec{w}^\perp$. This theorem is known as the **Orthogonal Decomposition Theorem**. Note: a corollary of this theorem is that $(\mathcal{W}^\perp)^\perp = \mathcal{W}$.

**EXAMPLE**

Consider the subspace $\mathcal{W}$, $x - y + 2z = 0$ with the vector $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Show that the orthogonal decomposition of $\vec{v}$ is $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$.

**Theorem 5.13**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ then $\dim(\mathcal{W}) + \dim(\mathcal{W}^\perp) = n$.

A corollary of Theorem 5.13 becomes clear when one applies it to the associated subspaces of a $m \times n$ matrix $A$. This is known as **The Rank Theorem**.

$\dim(\text{row}(A)) + \dim(\text{null}(A)) = n$ and $\dim(\text{col}(A)) + \dim(\text{null}(A^T)) = m$

**The Rank Theorem**

If $A$ is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$ and $\text{rank}(A) + \text{nullity}(A^T) = m$.

(Recall, $\text{rank}(A) = \text{rank}(A^T)$)