# Linear Systems

Math 214 Spring 2007 | Fowler 110 MWF 2:30pm - 3:25pm
 | http://faculty.oxy.edu/ron/math/214/07/

*Class 27*: **Friday April 6**

**TITLE** Orthogonal Complements and Orthogonal Projections
**CURRENT READING** Poole 5.1

**Summary**
We will learn about an incredibly important feature of vectors and orthogonal vector spaces.

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*Homework Assignment*
*HW#25 Poole, Section 5.2: 2,3,***4,5***,6,***7,12***,15,***16***,17,19,***20,21***. EXTRA CREDIT 29.*

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DEFINITION

Two subspaces $\mathcal{V}$ and $\mathcal{W}$ are said to be **orthogonal** if every vector $\vec{v} \in \mathcal{V}$ is perpendicular to every vector $\vec{w} \in \mathcal{W}$. The **orthogonal complement** of a subspace $\mathcal{V}$ contains EVERY vector that is perpendicular to (vectors in) $\mathcal{V}$. This space is denoted $\mathcal{V}^\perp$. In other words, $\vec{v} \cdot \vec{w} = 0$ or $\vec{v}^T\vec{w} = 0$ for every $\vec{v}$ in $\mathcal{V}$ and $\vec{w}$ in $\mathcal{W}$.

$$\mathcal{W}^\perp = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \text{ in } \mathcal{W}\}$$

*Example* 1. **Q:** In $\mathbb{R}^3$, let $V =$ the $z$-axis. What is $V^\perp$? **A:** ______________________

**Q:** In $\mathbb{R}^3$, what is the orthogonal complement of the $xy$-plane?
**A:** ______________________

**Q:** In $\mathbb{R}^3$, are the $xy$-plane and the $yz$-plane orthogonal complements of each other?
**A:** No, there are vectors in one plane that are not perpendicular to vectors in the other plane. (Can you find one of each?)

**Q:** In $\mathbb{R}^4$ (with axes $x_1, x_2, x_3, x_4$), what is the orthogonal complement of the $x_1x_2$-plane?

**A:** ______________________

We can summarize some of the properties of orthogonal complements.

**Theorem 5.9**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$.
[**a.**] $\mathcal{W}^\perp$ is a subspace of $\mathbb{R}^n$
[**b.**] $(\mathcal{W}^\perp)^\perp = \mathcal{W}$
[**c.**] $(\mathcal{W}^\perp) \cap \mathcal{W} = \vec{0}$
[**d.**] If $\mathcal{W}$=span$(\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n)$ then $\vec{v}$ is in $\mathcal{W}^\perp$ only if $\vec{v} \cdot \vec{w}_i = 0$ for every $\vec{w}_i$ in $\mathcal{W}$ for $i = 1 \ldots n$

These features can be described using the associated subspaces of an $m \times n$ matrix $A$.

**Theorem 5.10**

Let $A$ be an $m \times n$ matrix. Then the orthogonal complement of the row space of $A$ is the null space of $A$. The orthogonal complement of the column space of $A$ is the null space of $A^T$ (sometimes called the left null space). Mathematically, this can be written:

$$(\text{row}(A))^\perp = \text{null}(A) \text{ and } (\text{col}(A))^\perp = \text{null}(A^T)$$

These four subspaces are called the **fundamental subspaces of the matrix** $A$.

EXAMPLE

Let's find bases for the four fundamental subspaces of the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$.

Suppose we know that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Write down the dimensions of each fundamental subspace and describe the subspace-orthogonal complement pairs.

DEFINITION

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n\}$ be an orthogonal basis for $\mathcal{W}$. For any vector $\vec{v}$ in $\mathbb{R}^n$, the orthogonal project of $\vec{v}$ onto $\mathcal{W}$ is defined as

$$\text{proj}_{\mathcal{W}}(\vec{v}) = \sum_{j=1}^{n} \text{proj}_{\vec{w}_j}(\vec{v}) = \sum_{j=1}^{n} \frac{\vec{v} \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \vec{v}$$

The **component of $\vec{v}$ orthogonal to** $\mathcal{W}$ is the vector $\text{perp}_{\mathcal{W}}(\vec{v}) = \vec{v} - \text{proj}_{\mathcal{W}}(\vec{v})$

NOTE: this implies that $\vec{v} = \text{perp}_{\mathcal{W}}(\vec{v}) + \text{proj}_{\mathcal{W}}(\vec{v})$ (Draw a picture in $\mathbb{R}^2$!)

**Theorem 5.11**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $\vec{v}$ be ANY vector in $\mathbb{R}^n$. THEN there exist unique vectors $\vec{w}$ in $\mathcal{W}$ and $\vec{w}^{\perp}$ in $\mathcal{W}^{\perp}$ such that $\vec{v} = \vec{w} + \vec{w}^{\perp}$. This theorem is known as the **Orthogonal Decomposition Theorem**. Note: a corollary of this theorem is that $(\mathcal{W}^{\perp})^{\perp} = \mathcal{W}$.

EXAMPLE

Consider the subspace $\mathcal{W}$, $x - y + 2z = 0$ with the vector $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Show that the orthogonal decomposition of $\vec{v}$ is $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$

**Theorem 5.13**

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ then $\dim(\mathcal{W}) + \dim(\mathcal{W}^{\perp}) = n$.

A corollary of Theorem 5.13 becomes clear when one applies it to the associated subspaces of a $m \times n$ matrix $A$. This is known as **The Rank Theorem**.
$\dim(\text{row}(A)) + \dim(\text{null}(A)) = n$ and $\dim(\text{col}(A)) + \dim(\text{null}(A^T)) = m$

**The Rank Theorem**

If $A$ is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$ and $\text{rank}(A) + \text{nullity}(A^T) = m$.

(Recall, $\text{rank}(A) = \text{rank}(A^T)$)