Linear Systems

Math 214 Spring 2007 — Fowler 110 MWF 2:30pm - 3:25pm

 — http://faculty.oxy.edu/ron/math/214/07/

Class 26: **Wednesday April 4**

TITLE Orthogonality and Projections Revisited
CURRENT READING Poole 5.1

Summary
We shall return to the investigation of projections and orthogonality, this time with more increased generality.

Homework Assignment
*HW #24 Poole, Section 5.1: 3,4,5,6,8,***9,12***,13,16,***17,30,31***. EXTRA CREDIT 28, 33.*

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1. Orthogonal Bases

DEFINITION

An **orthogonal basis** of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is a basis of $\mathcal{W}$ that is an **orthogonal set** of vectors. An orthogonal set of vectors is a collection of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ where *every* pair of distinct vectors is orthogonal to each other, i.e. $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$.

Theorem 5.1

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ then those vectors are linearly independent.

EXAMPLE

Show that $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^3$.

Theorem 5.2

Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in $\mathcal{W}$. THEN the unique scalars $c_1, c_2, c_3, \ldots, c_n$ (also known as coordinates) where $\vec{w} = \sum_{i=1}^{n} c_i \vec{v}_i$ are given by

$$c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

EXAMPLE

Let's show how this formula for the coordinates is derived. (Doesn't it look familiar??)

Exercise

Given the orthogonal basis $\beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and the vector $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ find the coordinates of $\vec{w}$ with respect to β, i.e. $[\vec{w}]_\beta$.

DEFINITION

An **orthonormal basis** of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is a basis of $\mathcal{W}$ that consists of an **orthonormal set** of vectors. An orthonormal set of vectors is a collection of orthogonal unit vectors $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \dots, \vec{q}_k\}$ where $\vec{v}_i \cdot \vec{v}_j = \delta_{i,j}$. The symbol $\delta_{i,j}$ is known as the Kronecker delta function and has the property that $\delta_{i,j} = 0$ when $i \neq j$ and $\delta_{i,j} = 1$ when $i = j$.

Exercise

Form an orthonormal basis for $\mathbb{R}^3$ from the orthogonal basis β given in the previous **Exercise**.

2. Orthogonal Matrices

DEFINITION

A $n \times n$ matrix Q is said to be an **orthogonal matrix** if the columns (and rows) of the matrix form an orthonormal set.

Theorem 5.4

The columns of an $m \times n$ matrix Q form an orthonormal set if and only if $Q^T Q = I_n$.

Theorem 5.5

A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

Theorem 5.8

Let Q be an orthogonal matrix.

(a) Q^{-1} is orthogonal.

(b) $\det(Q) = \pm 1$.

(c) If λ is an eigenvalue of Q, then $|\lambda| = 1$.

(d) If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$.

EXAMPLE

Let's form a square orthogonal matrix from the orthonormal basis found in the previous exercise and illustrate some of the results from Theorem 5.4, 5.5 and 5.8.