Class 23: Wednesday March 28

TITLE Determinants

CURRENT READING Poole 4.3

Summary
Let’s explore the wonderful world of eigenvectors, eigenvalues and eigenspaces of a square \( n \times n \) matrix.

Homework Assignment
HW#21 Poole, Section 4.3: 4,5,10,15,16,17,18,20,21,23,33. EXTRA CREDIT 34,36,38.

DEFINITION
The eigenvalues of a square \( n \times n \) matrix \( A \) satisfy the characteristic polynomial of the matrix \( A \), given by \( \det(A - \lambda I) = 0 \).

EXAMPLE
Find the eigenvalues and corresponding eigenspaces of the matrix
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
2 & -5 & 4 \\
\end{pmatrix}
\]

DEFINITION
The algebraic multiplicity of an eigenvalue is the multiplicity of this eigenvalue as a root of the characteristic polynomial. The geometric multiplicity of an eigenvalue \( \lambda \) is the dimension of the corresponding eigenspace \( E_\lambda \), i.e. the number of vectors in a basis for the eigenspace.

Exercise
Write down the algebraic and geometric multiplicity of the eigenvalues of the matrix
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
2 & -5 & 4 \\
\end{pmatrix}
\]
Theorem 4.15
The eigenvalues of a triangular matrix (lower triangular, upper triangular or diagonal) are simply the entries along its main diagonal.

Theorem 4.16
Let $A$ be a square matrix with eigenvalue $\lambda$ and eigenvector $\vec{x}$
(i) For any integer $n$, $\lambda^n$ is an eigenvalue of $A^n$ with corresponding eigenvector $\vec{x}$
(ii) If $A$ is invertible, then $1/\lambda$ is an eigenvalue of $A^{-1}$ with corresponding eigenvector $\vec{x}$

Theorem 4.18
A square matrix $A$ is invertible if and only if $0$ is NOT an eigenvalue of $A$.

**EXAMPLE**
Poole, page 296, #19. (a) Show that for any square matrix $A$, $A^T$ and $A$ have the same characteristic polynomial and thus the same eigenvalues.

(b) Give an example of a $2 \times 2$ matrix $A$ for which $A^T$ and $A$ have different eigenspaces.

**Exercise**
Show that the eigenvalues $A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$ are 5 and $-2$ and $E_{-2} = \text{span} \left( \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right)$ and $E_{5} = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

Find the eigenvalues of $3A$, $A^{-1}$, $A^2$ and $A + I$
Linear Independence of Eigenvectors

**Theorem 4.19**
Suppose the $n \times n$ matrix $A$ has $m$ eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. IF $\vec{x}$ is a vector in $\mathbb{R}^n$ that can be written as a linear combination of these vectors, THEN

$$A^k \vec{x} = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2 + c_3 \lambda_3^k \vec{v}_3 + \ldots + c_m \lambda_m^k \vec{v}_m$$

**EXAMPLE**
Let’s use this result to show that

\[
\begin{bmatrix}
3 & 2 \\
5 & 0
\end{bmatrix}^6 \begin{bmatrix}
1 \\
8
\end{bmatrix} = \begin{bmatrix}
46747 \\
47195
\end{bmatrix}
\]

**Theorem 4.20**
Let $A$ be an $n \times n$ matrix with $m$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$. Then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$ are linearly independent.

**Properties of the Eigenvalues of a $n \times n$ Matrix**

The **Product** of the eigenvalues equals the determinant of the $n \times n$ matrix.

$$\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n = |A|$$

The **Sum** of the eigenvalues equals the trace of the $n \times n$ matrix (the sum of the diagonal entries)

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \sum_{i=1}^{n} A_{ii}$$