Linear Systems

Class 22: **Monday March 26**

TITLE Determinants
CURRENT READING Poole 4.2

Summary
Let's understand how to compute the determinant of a $n \times n$ matrix and understand the properties and applictations of determinants.

Homework Assignment
HW #20 Poole, Section 4.2: 4,6, 7, **10**, *15, 26, 27, 32,* **33, 48, 49,50,51**, *52. EXTRA CREDIT 46.*

DEFINITION

Let A be any matrix. The ij-**minor** of A is the matrix obtained by removing its ith row and its jth column. It is denoted by $\hat{A}_{ij}$.

Theorem 4.1

Let A be any n by n matrix. The **determinant** of A is defined as:

If $n = 1$, then $|A| = A_{11}$.

If $n \geq 2$, then

$|A| = A_{11}|\hat{A}_{11}| - A_{12}|\hat{A}_{12}| + \cdots + A_{1n}|\hat{A}_{1n}|$

More general definition: Fix any row i. Then,

$$|A| = \sum_{j=1}^{n} A_{ij}(-1)^{i+j}|\hat{A}_{ij}| = \sum_{j=1}^{n} A_{ij}C_{ij}$$

Or, fix any column j. Then,

$$|A| = \sum_{i=1}^{n} A_{ij}(-1)^{i+j}|\hat{A}_{ij}| = \sum_{i=1}^{n} A_{ij}C_{ij}$$

Although it may seem like it, this definition is not really a "circular" definition.
It's known as a *recursive* definition. The formula is called the **Laplace Expansion Theorem**. The number $C_{ij} = (-1)^{i+j}|\hat{A}_{ij}|$ is known as the i,j-**cofactor** of A.

EXAMPLE

Let's compute the determinant of $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Ans: $\begin{vmatrix} 1 & 4 & 2 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1(10-1) - 4(6-0) + 2(3-0) = -9$

Exercise

Compute the determinant of the matrix using a different row and column

1. Properties of the Determinant of a Matrix

Here is a summary of the various properties of the determinant.

1. The determinant is a linear function of the first row.
2. The determinant changes sign when two rows are exchanged.
3. The determinant of the n by n identity matrix is 1.
4. If two rows of A are equal, then $\det A = 0$.
5. Subtracting a multiple of one row from another row leaves $\det A$ unchanged.
6. A matrix with a row of zeros has $\det A = 0$.
7. If A is a triangular matrix the $\det A$ equals the product of the main diagonal entries.
8. If A is singular then $\det A = 0$. If A is invertible then $\det A \neq 0$.
9. The determinant of AB is the product of the separate determinants: $|AB| = |A||B|$.
10. The transpose A^T has the same determinant as A.

GroupWork

Come up with examples to illustrate principles (4) - (10) above.

2. Applications of the Determinant

Theorem 4.7

If A is a n by n matrix then $|kA| = k^n|A|$

Theorem 4.9

If A is an invertible n by n matrix then $|A^{-1}| = \dfrac{1}{|A|}$

EXAMPLE

The sum of the determinant of two matrices is **NOT** equal to the determinant of the sum of two matrices, i.e. $|A + B| \neq |A| + |B|$. Let's prove this.

Cramer's Rule

Suppose we're given a linear system of equations $A\vec{x} = \vec{b}$, where A and $\vec{b}$ are given, and we are to find $\vec{x}$. We have learned how to solve this using Gaussian Elimination. A *longer* way to find $\vec{x}$ is as follows!

Theorem (Cramer's Rule) Given $A\vec{x} = \vec{b}$, the jth coordinate of $\vec{x}$ is given by the formula

$$x_j = \frac{\det(B_j)}{\det(A)}$$

where B_j is obtained by replacing the j-th column of A by $\vec{b}$.

Example 1. Solve the system $\begin{aligned} 2x + 4y &= 1 \\ x + 3y &= 2 \end{aligned}$ using Cramer's Rule.

Ans:

Cramer's Rule takes *a lot more* work than Gaussian Elimination to solve a system. So why is it useful? I think because it gives us a *formula* for the solution, as opposed to Elimination, which is only a procedure for finding the solution.

Formula for A^{-1}

Theorem If A is an invertible matrix, then A^{-1} is given by

$$(A^{-1})_{i,j} = \frac{C_{j,i}}{\det(A)}$$

where $C_{j,i}$ is the cofactor of $A_{j,i}$.

Find the inverse of $\begin{bmatrix} 2 & 6 & 2 \\ 0 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}$ using the co-factor formula

The vector cross product

Definition 1. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Their **cross product** is defined as $\vec{u}\times\vec{v} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$.

Note. $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, and $\hat{k} = (0,0,1)$. These are unit vectors. But in computing the above determinant, just treat them as symbols or numbers. **Note:** the result of a cross-product is a vector which is orthogonal (perpendicular) to both vectors in the product. (Where could this be useful?)

Use the cross-product to find the general equation $Ax + By + Cz = D$ of the plane in $\mathbb{R}^3$ which is the span $\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right)$