Linear Systems

Math 214 Spring 2007 Fowler 110 MWF 2:30pm - 3:25pm
 http://faculty.oxy.edu/ron/math/214/07/

Class 19: **Monday March 19**

TITLE Introduction to Linear Transformations
CURRENT READING Poole 3.6

Summary
We'll introduce the concept of linear transformation. This is a very cool, visually interesting application of linear algebra.

Homework Assignment
*HW 17: Poole, Section 3.6: 1,***2***,3,***6***,***7***,15,17,***19***. EXTRA CREDIT 28.*

Warm-Up
What's a function (broadly defined)? Can you think of function which has a $n \times n$ matrix as its input and a number as an output? What about a function which has a 5×1 vector as input and a 3×1 vector as output?

DEFINITION
A **transformation** or **function** or **mapping** from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{v} \in \mathbb{R}^n$ a unique vector $T(\vec{v}) \in \mathbb{R}^m$. The **domain** of T is $\mathbb{R}^n$ and the **co-domain** is $\mathbb{R}^m$. This is denoted $T : \mathbb{R}^n \to \mathbb{R}^m$. Given a vector $\vec{v}$ in the domain of T, the vector $T(\vec{v})$ is called the **image** of $\vec{v}$ under the action of T. The set of all possible images $T(\vec{v})$ is called the **range** of T.

DEFINITION
A transformation (or mapping or function) $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars c.

EXAMPLE
Given $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$. If we multiply A by an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$ we can define a transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

What is the image of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$? What is the pre-image of $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$? What is the range of T_A? Is T_A a linear transformation?

Theorem 3.30

Given a $m\times n$ matrix, the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\vec{x}) = A\vec{x}$ (for all $\vec{x}$ in $\mathbb{R}^n$) is a linear transformation.

Theorem 3.31

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as a matrix transformation T_A where A is the $m\times n$ matrix whose columns are given by the images of the standard basis vectors in $\mathbb{R}^n$ under the action of T, i.e.

$$A = \begin{bmatrix} T(\hat{e}_1) & T(\hat{e}_2) & T(\hat{e}_3) & \ldots & T(\hat{e}_n) \end{bmatrix}$$

This matrix is called the **standard matrix of the linear transformation** T and can be denoted $[T]$.

Exercise

Find the standard matrix for the linear transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ which maps each point to its reflection in the x-axis.

Find the standard matrix for the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each point 90^o counterclockwise about the origin.

EXAMPLE

Let's find the standard matrix for the linear transformation R_θ which rotates a point θ degrees counterclockwise about the origin.

Composition of Linear Transformations

Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^p$ are linear transformations. Then the **composition** of the two transformations is denoted $S \circ T$. This means that a vector in the domain of T is mapped into the co-domain of S. The interpretation of the composition is a vector $\vec{v}$ acted on by T which is then acted on by S, i.e. $(S \circ T)(\vec{v}) = S(T(\vec{v}))$

Theorem 3.32

The standard matrix of a composition of linear transformations is equal to the product of the standard matrices of each transformation. $[S \circ T] = [S][T]$

EXAMPLE

Let's find the standard matrix of the linear transformation that first rotates a point 90^o about the origin and then reflects the result in the x-axis.

DEFINITION

The **identity transformation** is the transformation $I : \mathbb{R}^n \to \mathbb{R}^n$ which leaves every vector unchanged. If S and T are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ so that $S \circ T = I$ and $T \circ S = I$ then S and T are **inverse transformations** of each other. Both S and T are said to be invertible linear transformations.

Theorem 3.33

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then the standard matrix $[T]$ is an invertible matrix, and $[T^{-1}] = [T]^{-1}$. In other words, the standard matrix for the inverse linear transformation of T is the inverse of the standard matrix for the linear transformation T.

Exercise

Show that the linear transformation which maps a point θ degrees counterclockwise in $\mathbb{R}^2$ about the origin is **the inverse** of the linear transformation which maps a point θ degrees *clockwise*.