Class 19: Monday March 19

TITLE Introduction to Linear Transformations

CURRENT READING Poole 3.6

Summary
We’ll introduce the concept of linear transformation. This is a very cool, visually interesting application of linear algebra.

Gateway Assignment
HW 17: Poole, Section 3.6: 1, 2, 3, 6, 7, 15, 17, 19. EXTRA CREDIT 28.

Warm-Up
What’s a function (broadly defined)? Can you think of function which has a $n \times n$ matrix as its input and a number as an output? What about a function which has a $5 \times 1$ vector as input and a $3 \times 1$ vector as output?

**DEFINITION**
A transformation or function or mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{v} \in \mathbb{R}^n$ a unique vector $T(\vec{v}) \in \mathbb{R}^m$. The domain of $T$ is $\mathbb{R}^n$ and the co-domain is $\mathbb{R}^m$. This is denoted $T : \mathbb{R}^n \to \mathbb{R}^m$. Given a vector $\vec{v}$ in the domain of $T$, the vector $T(\vec{v})$ is called the image of $\vec{v}$ under the action of $T$. The set of all possible images $T(\vec{v})$ is called the range of $T$.

**DEFINITION**
A transformation (or mapping or function) $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars $c$.

**EXAMPLE**
Given $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$. If we multiply $A$ by an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$ we can define a transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

What is the image of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$? What is the pre-image of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$? What is the range of $T_A$? Is $T_A$ a linear transformation?
**Theorem 3.30**
Given a $m \times n$ matrix, the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\vec{x}) = A\vec{x}$ (for all $\vec{x}$ in $\mathbb{R}^n$) is a linear transformation.

**Theorem 3.31**
Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as a matrix transformation $T_A$ where $A$ is the $m \times n$ matrix whose columns are given by the images of the standard basis vectors in $\mathbb{R}^n$ under the action of $T$, i.e.

$$A = \begin{bmatrix}
T(\hat{e}_1) & T(\hat{e}_2) & T(\hat{e}_3) & \ldots & T(\hat{e}_n)
\end{bmatrix}$$

This matrix is called the **standard matrix of the linear transformation** $T$ and can be denoted $[T]$.

**Exercise**
Find the standard matrix for the linear transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ which maps each point to its reflection in the $x$-axis.

Find the standard matrix for the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each point $90^\circ$ counterclockwise about the origin.

**EXAMPLE**
Let’s find the standard matrix for the linear transformation $R_\theta$ which rotates a point $\theta$ degrees counterclockwise about the origin.
Composition of Linear Transformations
Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^p$ are linear transformations. Then the composition of the two transformations is denoted $S \circ T$. This means that a vector in the domain of $T$ is mapped into the co-domain of $S$. The interpretation of the composition is a vector $\vec{v}$ acted on by $T$ which is then acted on by $S$, i.e. $(S \circ T)(\vec{v}) = S(T(\vec{v}))$

**Theorem 3.32**
The standard matrix of a composition of linear transformations is equal to the product of the standard matrices of each transformation. $[S \circ T] = [S][T]$

**EXAMPLE**
Let's find the standard matrix of the linear transformation that first rotates a point $90^\circ$ about the origin and then reflects the result in the $x$-axis.

**Definition**
The identity transformation is the transformation $I : \mathbb{R}^n \to \mathbb{R}^n$ which leaves every vector unchanged. If $S$ and $T$ are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ so that $S \circ T = I$ and $T \circ S = I$ then $S$ and $T$ are inverse transformations of each other. Both $S$ and $T$ are said to be invertible linear transformations.

**Theorem 3.33**
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then the standard matrix $[T]$ is an invertible matrix, and $[T^{-1}] = [T]^{-1}$. In other words, the standard matrix for the inverse linear transformation of $T$ is the inverse of the standard matrix for the linear transformation $T$.

**Exercise**
Show that the linear transformation which maps a point $\theta$ degrees counterclockwise in $\mathbb{R}^2$ about the origin is the inverse of the linear transformation which maps a point $\theta$ degrees clockwise.