**Class 14: Monday February 26**

**SUMMARY**  LU Decomposition and Permutation Matrices

**CURRENT READING**  Poole 3.4

**Summary**
We have found that we could (sometimes) find a matrix $A^{-1}$ which converted $A$ into the identity matrix $I$, on multiplication. We had also previously shown that we could find a series of $E_{ij}$ matrices which when multiplied in sequence would convert $A$ into an upper triangular matrix. Today we will attempt to regularize this process and show we can use these ideas to factor a matrix $A$ into the product of a lower triangular matrix $L$ and upper triangular matrix $U$.

**Homework Assignment**
HW # 14: Section 3.4: 1,2,3,7,8,9,10,13,19,20. EXTRA CREDIT 26.

1. **LU Factorization**

Consider the matrix $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$

Can you show that this can be converted into upper triangular form by multiplying by a series of matrices $E_{21}$, $E_{31}$ and $E_{32}$?

$U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

We have that $E_{32}E_{31}E_{21}A = U$

This means that

$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = L \cdot U$
Write down the elimination matrices you used to convert $A$ into $U$

Write down the INVERSE of each of these three matrices.

Note that all of these matrices $E_{21}$, $E_{31}$, $E_{32}$, $E_{21}^{-1}$, $E_{31}^{-1}$ and $E_{32}^{-1}$ are all LOWER TRIANGULAR.

Compute the product $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. It is ALSO lower triangular. We call it $L$.

Now check that the product of $L$ and $U$ is, in fact, $A$. 
The Point
We can use $LU$ factorization to assist us in solving $A\vec{x} = \vec{b}$

$L U \vec{x} = \vec{b}$ becomes the two systems of $L \vec{c} = \vec{b}$ and $U \vec{x} = \vec{c}$

Solving a lower triangular system and then an upper triangular system is much more computationally efficient than finding $A^{-1}$.

Let’s do an example with $\vec{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ and our given $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$
2. Permutation Matrices An $n \times n$ permutation matrix $P$ has the rows of the $n \times n$ identity matrix $I$ in any order. In other words, it has exactly one 1 in each row and column.

Clearly, there are $n!$ permutation matrices of order $n$. (Think about how you would prove this.)

Permutation matrices have the property that $P^T = P^{-1}$.

**Group Work**

Write down the $2!$ matrices of order 2 (i.e. of dimension $2 \times 2$)

Write down the $3!$ matrices of order 3

**Exercise**

Choose any one of your permutation matrices of order 3 from above and confirm that it has the property that $P^T = P^{-1}$. 