Class 12: Wednesday February 21

SUMMARY The Inverse Matrix
CURRENT READING Poole 3.3

Summary
We will introduce a very important concept, the Inverse Matrix.

Homework Assignment
HW # 12: Section 3.3 # 2,5, 9,10,19,20,21, 22,23 EXTRA CREDIT # 13

1. Inverse Matrix
Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Write down the product of $M$ and $A$. That is, $MA$ and $AM$.

We call $M$, the matrix which when multiplied by $A$ produces the identity matrix, the inverse matrix. It is denoted $A^{-1}$.

It has the property that $A^{-1}A = AA^{-1} = I$

The factor $ad-bc$ is known as the determinant of the matrix $A$. We will learn more about how to compute determinants and their significance later. However, it is true that if the determinant of a matrix equals zero, then that matrix is NOT invertible, i.e. $\det A = 0 \Rightarrow A^{-1}$ doesn’t exist. It is also true that if $A^{-1}$ doesn’t exist $\Rightarrow \det(A) = 0$.

**Theorem 3.6**
If $A$ is an invertible matrix, then its inverse $A^{-1}$ is unique.

**Theorem 3.7**
If $A$ is an invertible $n \times n$ matrix, then the system of linear equations given by $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$ for any $\vec{b}$ in $\mathbb{R}^n$. 
2. Computing Inverses: Gauss-Jordan Elimination

In order to actually generate or find an inverse matrix we use a process called Gauss-Jordan elimination. This is identical to the Gaussian elimination process we already know, except extended.

Consider the system

\[
\begin{align*}
1x + 2y - 1z &= 1 \\
2x + 2y + 4z &= 3 \\
1x + 3y - 3z &= 0
\end{align*}
\]

Write down the augmented matrix with the identity matrix as the right hand side.

\[
\begin{bmatrix}
1 & 2 & -1 & | & 1 & 0 & 0 \\
2 & 2 & 4 & | & 0 & 1 & 0 \\
1 & 3 & -3 & | & 0 & 0 & 1
\end{bmatrix}
\]

We will do Gaussian Elimination on this system until we have produced the identity matrix on the left 3x3 matrix.
3. Properties of Inverses

(1) \((A^{-1})^{-1} = A\)
(2) \((AB)^{-1} = B^{-1}A^{-1}\)
(3) \((ABC)^{-1} = C^{-1}B^{-1}A^{-1}\)
(4) \((A^{-1})^n = (A^n)^{-1}\) for positive integers \(n\)
(5) \((A^{-1})^T = (A^T)^{-1}\)
(6) \(\frac{1}{c}A^{-1} = (cA)^{-1}\) for positive scalars \(c \neq 0\)

Exercise

Consider \(A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\) and \(B = \begin{bmatrix} -5 & -7 \\ 2 & 3 \end{bmatrix}\). Show that \(A^{-1}B^{-1} = (BA)^{-1}\).

4. Determining Singularity

If the determinant of a coefficient matrix is zero, then the system is singular (no solution or infinite number of solutions) and thus the linear system can not be solved.

\[
\det(A) = 0 \iff A^{-1}\text{ doesn’t exist}
\]

So it is NOT always possible to find \(A^{-1}\). \(A^{-1}\) exists ONLY IF a \(n \times n\) matrix \(A\) has rank\((n)\).
Using Gauss-Jordan To Solve Linear Systems

Gauss-Jordan takes the augmented matrix \([ A|I ]\) and converts it into \([ I|A^{-1} ]\).

Q: What has happened to each block matrix in the augmented matrix?
A: Each block matrix been multiplied by by \(A^{-1}\).

Therefore Gauss-Jordan can also take the matrix \([ A|I|\vec{b} ]\) and convert into \([ I|A^{-1}|A^{-1}\vec{b} ]\)

Why is this useful?

Gauss-Jordan works by solving \(n\) linear systems at once.
For a 3x3 system it is solving \(A\vec{x}_1 = \vec{e}_1\), \(A\vec{x}_2 = \vec{e}_2\) and \(A\vec{x}_3 = \vec{e}_3\)

where \(\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\), \(\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\) and \(\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\).

The vectors \(\vec{x}_1\), \(\vec{x}_2\) and \(\vec{x}_3\) which solve the 3 equations above are simply the columns of the inverse matrix.

Example
Consider the system (with \(d \neq 0\))

\[
\begin{bmatrix}
1 & 1 & 1 & | & 1 & 0 & 0 & 1 \\
1 & (d+1) & 3 & | & 0 & 1 & 0 & 5 \\
0 & 2 & d & | & 0 & 0 & 1 & -4 \\
\end{bmatrix}
\]

Let’s use Gauss-Jordan to find the solution