Linear Systems

Math 214 Spring 2007 — Fowler 110 MWF 2:30pm - 3:25pm

 — http://faculty.oxy.edu/ron/math/214/07/

Class 10: **Wednesday February 14**

SUMMARY Matrix Properties
CURRENT READING Poole 3.1

Summary
We begin our study of Chapter 3 by considering matrices as objects in their own right, and not just as ways of viewing, or parts of, linear systems.

Homework Assignment
*HW # 10: Section 3.1: 1,2,3,4,5,6,***7, 8,34,35***. EXTRA CREDIT 37: DUE FRI FEB 16*

1. Matrix Definitions

DEFINITION

Let A be an $m \times n$ **matrix** (with m rows and n columns). If $m = n$, then A is said to be a **square matrix**. For $1 \le i \le m$ and $1 \le j \le n$, the (i, j)**-entry** of A, denoted by $A_{i,j}$, is the number in the ith row and the jth column of A. We denote the ith row of A by $\mathbf{row}_i(A)$, and the jth column of A by $\mathbf{col}_j(A)$.

Note. For convenience, some books, including ours, drop the comma from $A_{i,j}$, and instead write A_{ij}. You may do this too, except when it can cause ambiguity, as in: $A_{123} = A_{12,3}$ or $A_{1,23}$?

Q: An m-component column vector is a ?×? matrix? **A:**

Q: An n-component row vector is a ?×? matrix? **A:**

DEFINITION

Let A and B be $m \times n$ matrices. Then their **sum** $A+B$ is an $m \times n$ matrix C defined by: $C_{i,j} = A_{i,j} + B_{i,j}$.

Example 1. Compute $B + A$, where $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

DEFINITION

Let A and B be $m \times n$ matrices. Then A is said to be equal to B if both A and B have the same dimensions **and** if $A_{i,j} = B_{i,j}$ for every i and j in each matrix.

Example 2. **Q:** Are $\begin{bmatrix} 1 & 0 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ equal? **A:** No! (Why not?)

DEFINITION

Let A be a $m \times n$ matrix and c is a real number. Then cA is said to a scalar multiple of A and cA is obtained by multiplying each element of A by c.

DEFINITION

Let O be a $m \times n$ matrix called the **zero matrix** where every entry equals zero. Clearly, $A + O = O + A = A$ and $A - A = -A + A = O$. The zero matrix acts like the matrix "additive identity" also known as the number "zero."

2. Matrix Multiplication

We add matrices component-wise: $(A+B)_{i,j} = A_{i,j} + B_{i,j}$. But we do not multiply matrices component-wise: $(AB)_{i,j} \neq A_{i,j}B_{i,j}$ (just as vector addition is component-wise, but the dot product isn't).

DEFINITION

Let A be an $m \times n$ matrix, and B an $n \times q$ matrix. Then their **product** AB is an $m \times q$ matrix C defined by $C_{i,j} = \text{row}_i(A) \cdot \text{col}_j(B)$. (Equivalently, C can be defined by: $\text{col}_j(C) = A\,\text{col}_j(B)$.)

Example 3. Compute BA, where $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Q: What type of matrix can be multiplied by itself? **Ans:** A square matrix.

Notation: $AA = A^2, AAA = A^3, \cdots$. Also, note that $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$ when r and s are non-negative integers.

Example 4. Compute A^2 and A^3

DEFINITION

The $n \times n$ **identity matrix** I or I_n is a square matrix defined to have 1's along its diagonal, and 0's elsewhere. The identity matrix acts like the matrix "multiplicative identity" also known as the number "one." Clearly, $AI = IA = A$.

DEFINITION

Two $n \times n$ matrices A and B are said to be **inverses** of each other if $AB = I_n$ and $BA = I_n$.

3. Matrix Transposes

DEFINITION

Given a matrix A, the transpose matrix is denoted A^T. The rows of A become the columns of A^T. If A is $m \times n$ then A^T is $n \times m$. Specifically, $A^T_{ij} = A_{ji}$.

4. Properties of the Transpose

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(cA)^T = cA^T$
- $(A^r)^T = (A^T)^r$ for non-negative integers r
- Recall that $A\vec{x}$ is a linear combination of the **columns** of A, so $x^T A^T$ is a linear combination of the ROWS of A^T

Exercise

Confirm the above transpose properties with $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

5. Symmetric Matrices

DEFINITION

A matrix is said to be *symmetric* if it is its own transpose, i.e. $A^T = A$.

The inverse of a symmetric matrix is also symmetric.

The product of a matrix with its transpose produces a symmetric matrix.
$R^T R = R^T (R^T)^T = R^T R$

6. Block Matrices

Consider $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$

Exercise

Write down AB in terms of the elements of A and B.

Now, suppose the elements of A and B are themselves matrices!

$A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A_{12} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 4 & 0 \end{bmatrix}$ and $A_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A_{22} = \begin{bmatrix} 1 & 7 \\ 7 & 2 \end{bmatrix}$

$B_{11} = \begin{bmatrix} 4 & 3 \\ -1 & 2 \\ 1 & -5 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}$, $B_{13} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B_{23} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Now compute AB block by block. First check that matrix A and B are **partitioned conformably for block multiplication**. (In other words, that in every possibly matrix multiplication the dimensions match up properly.)