Linear Systems

Class 4: **Monday January 29**

SUMMARY Equations of Planes
CURRENT READING Poole 1.3

OUTLINE
We will extend our examination of the multiple versions of equations for lines to the corresponding equations for planes. We will continue with numerous examples to illustrate our understanding of the analytic geometry of planes and lines using formulas for distance, projections and angles.

Homework Assignment
HW #4: Section 1.3: 1, 4, 5, **11, 12, 20***: DUE WED FEB 1*

Equations of a Plane in $\mathbb{R}^3$

The main way we often think of planes in euclidean space is to define a plane in $\mathbb{R}^3$ as that two-dimensional object consisting of the set of points which contains the point $\vec{p}$ and are also exactly perpendicular to a particular vector $\hat{n}$:

General Form of the Equation of a Plane in $\mathbb{R}^3$

The **general form** of the equation of a plane L in $\mathbb{R}^3$ is $ax + by + cz = d$. In this case the vector $\hat{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a **normal vector** to the plane P.

Normal Form of the Equation of a Plane in $\mathbb{R}^3$

The **normal form** of the equation of a plane P in $\mathbb{R}^3$ is $\hat{n} \cdot (\vec{x} - \vec{p}) = 0$ or $\hat{n} \cdot \vec{x} = \hat{n} \cdot \vec{p}$. In this case the non-zero vector $\hat{n}$ is again a **normal vector** to the plane P and $\vec{p}$ is a particular given point on the plane P.

Vector Form of the Equation of a Plane in $\mathbb{R}^3$

The **vector form** of the equation of a plane P in $\mathbb{R}^3$ (or $\mathbb{R}^n$) is $\vec{x} = \vec{p} + t\vec{v} + s\vec{u}$. In this case the vectors $\vec{u}$ and $\vec{v}$ are not zero, not parallel to each other but are parallel to the plane P and $\vec{p}$ is a particular given point on the plane.

Parametric Form of the Equation of a Plane in $\mathbb{R}^3$

The **parametric form** of the equation of a plane P in $\mathbb{R}^3$ is the set of equations formed from the three components of the **vector form** of the equation of the plane. In this case those equations are $x = p_1 + v_1 t + u_1 s$, $y = p_2 + v_2 t + u_2 s$ and $z = p_3 + v_3 t + u_3 s$ where $\vec{p} = [p_1, p_2, p_3]$ and $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$. Note, if the vector has n components, then the parametric form of P will consist of n linear equations in the parametric variables s and t.

Dimension and Parameters

Intuitively, we understand that planes are **2-dimensional** objects and lines are **1-dimensional** objects. It is NOT a coincidence that equations of planes, particularly in vector or parametric form require **TWO** unknown variables (i.e. s and t) and equations of a line require **ONE** unknown variable, t.

Equations of a Line in $\mathbb{R}^3$

Lines in $\mathbb{R}^3$ are best thought of as the intersection of two non-parallel planes, or you can think of them as an object which is perpendicular to **two** vectors simultaneously and goes through a given point.

Below is a table (reproduced from **page 38 of Poole**) which summarizes the **normal**, **general**, **vector** and **parametric** forms of the equation of a **line** (and of a plane) in $\mathbb{R}^3$.

	Normal Form	General Form	Vector Form	Parametric Form
LINES	$\hat{n}_1 \cdot \vec{x} = \hat{n}_1 \cdot \vec{p}_1$ $\hat{n}_2 \cdot \vec{x} = \hat{n}_2 \cdot \vec{p}_2$	$a_1x + b_1y + c_1z = d_1$ $a_2x + b_2y + c_2z = d_2$	$\vec{x} = \vec{p} + t\vec{d}$	$x = p_1 + td_1$ $y = p_2 + td_2$ $z = p_3 + td_3$
PLANES	$\hat{n} \cdot \vec{x} = \hat{n} \cdot \vec{p}$	$ax + by + cz = d$	$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$	$x = p_1 + su_1 + tv_1$ $y = p_2 + su_2 + tv_2$ $z = p_3 + su_3 + tv_3$

More Thoughts On Dimension

Recall, in $\mathbb{R}^2$ the single general linear equation $ax + by = c$, or more generally $ax_1 + bx_2 = c$, represents a ______________________________. This is a 1-dimensional object.

In $\mathbb{R}^3$, the single general linear equation is $ax + by + cz = d$, or more generally $ax_1 + bx_2 + cx_3 = d$, represents a ______________________________. This is a **2-dimensional** object.

In $\mathbb{R}^b$, the single general linear equation is $k_1x_1 + k_2x_2 + k_3x_3 + \ldots k_nx_n = k$. This is a $n-1$**-dimensional** object, called a **hyperplane** in $\mathbb{R}^n$.

The point here is the relationship between the number of equations it takes to describe an object, the dimension of the object and the dimension of the space it is in. The relationship is

dimension of the object + number of equations = dimension of the space

Dimension will be an important topic we won't really formally define until much later in the class.

Let's do some example analytic geometry problems involving lines and planes to test our understanding of the concepts.

Exercise

Poole, Page 56, #10. Find the general equation of the plane through the points $A(1, 1, 0)$, $B(1, 0, 1)$ and $C(0, 1, 2)$.

EXAMPLE

Poole, Page 56, #9. Find the general equation of the plane through the point $(3, 2, 5)$ that is parallel to the plane whose general equation is $2x + 3y - z = 0$.

Exercise

Poole, Page 43, #43. Find the acute angle between the planes $x + y + z = 0$ and $2x + y - 2z = 0$.

EXAMPLE

Poole, Page 43, #37. Find the distance between the planes $2x + y - 2z = 0$ and $2x + y - 2z = 5$.