Linear Systems

Math 214 Spring 2006 — Fowler 307 MWF 2:30pm - 3:25pm

 — http://faculty.oxy.edu/ron/math/214/06/

Class 32: **Wednesday April 26**

TITLE Singular Value Decomposition
CURRENT READING Poole 7.4

Summary
Our final new topic will be the singular value decomposition of a matrix.

Homework Assignment
*Poole, Section 7.4 : 2,***5,6,7,13,45***,46,47.*

DEFINITION

The **singular values** of a $m \times n$ matrix A are the square roots of the eigenvalues of $A^T A$ and are usually denoted by $\sigma_1, \sigma_2, \sigma_3 \ldots \sigma_n$. Often the singular values are arranged so that $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_n$.

Theorem 7.13

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \ldots = \sigma_n = 0$. THEN there exists an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and a block-diagonal $m \times n$ matrix Σ such that $A = U \Sigma V^T$.

The columns of U are called the left singular vectors of A and the columns of V are called the right singular vectors of A. U and V are NOT uniquely determined by A but Σ must contain the singular values of A.

EXAMPLE

Find the singular value decomposition for the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem 7.15

Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix A. Let $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_r$ be all the nonzero singular values of A. THEN

(a) The rank of A is r.

(b) The first r columns of U, $[\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r]$ form an orthonormal basis for col(A).

(c) The last $m - r$ columns of U, $[\vec{u}_{r+1}, \vec{u}_{r+2}, \ldots, \vec{u}_m]$ form an orthonormal basis for null(A^T).

(d) The first r columns of V, $[\vec{v}_1, \vec{v}_2, \ldots, \vec{u}_r]$ form an orthonormal basis for row(A).

(e) The last $n - r$ columns of V, $[\vec{v}_{r+1}, \vec{v}_{r+2}, \ldots, \vec{v}_n]$ form an orthonormal basis for null(A).

Pretty cool, eh?

Notice also, that by definition $A\vec{v}_i = \sigma_i \vec{u}_i$ for $i = 1$ to r. So, singular values are "eigenvalue-like." There are similarities and differences between eigenvalues and singular values, however.

GROUPWORK

Consider A=$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$. What important property do you notice about A?

Find the eigenvalues of A.

Find the singular values of A (i.e. the positive square roots of the eigenvalues of $A^T A$).

What do you notice about the answers to the two previous questions? Explain your observation.

Recall that if $A \sim B$ then the eigenvalues of A and B are equal. This is NOT true for singular values.

Recall that the determinant of a matrix A is equal to the product of the eigenvalues; it is also equal to **the product of the singular values of A.**