Class 27: Monday April 10

TITLE Orthogonal Complements and Orthogonal Projections
CURRENT READING Poole 5.1

Summary
We will learn about an incredibly important feature of vectors and orthogonal vector spaces.

Homework Assignment
Poole, Section 5.2: 2, 3, 4, 5, 6, 7, 12, 15, 16, 17, 19, 20, 21. EXTRA CREDIT 29.

DEFINITION
Two subspaces \( V \) and \( W \) are said to be orthogonal if every vector \( \vec{v} \in V \) is perpendicular to every vector \( \vec{w} \in W \). The orthogonal complement of a subspace \( V \) contains EVERY vector that is perpendicular to (vectors in) \( V \). This space is denoted \( V^\perp \). In other words, \( \vec{v} \cdot \vec{w} = 0 \) or \( \vec{v}^T \vec{w} = 0 \) for every \( \vec{v} \) in \( V \) and \( \vec{w} \) in \( W \).

\[
W^\perp = \{ \vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \}
\]

Example 1. Q: In \( \mathbb{R}^3 \), let \( V \) = the z-axis. What is \( V^\perp \)? A: 

Q: In \( \mathbb{R}^3 \), what is the orthogonal complement of the xy-plane?
A: 

Q: In \( \mathbb{R}^3 \), are the xy-plane and the yz-plane orthogonal complements of each other?
A: No, there are vectors in one plane that are not perpendicular to vectors in the other plane. (Can you find one of each?)

Q: In \( \mathbb{R}^4 \) (with axes \( x_1, x_2, x_3, x_4 \)), what is the orthogonal complement of the \( x_1x_2 \)-plane?
A: 

We can summarize some of the properties of orthogonal complements.

**Theorem 5.9**
Let \( W \) be a subspace of \( \mathbb{R}^n \).

[a.] \( W^\perp \) is a subspace of \( \mathbb{R}^n \)

[b.] \( (W^\perp)^\perp = W \)

[c.] \( (W^\perp) \cap W = \vec{0} \)

[d.] If \( W = \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n) \) then \( \vec{v} \) is in \( W^\perp \) only if \( \vec{v} \cdot \vec{w}_i = 0 \) for every \( \vec{w}_i \) in \( W \) for \( i = 1 \ldots n \)

These features can be described using the associated subspaces of an \( m \times n \) matrix \( A \).

**Theorem 5.10**
Let \( A \) be an \( m \times n \) matrix. Then the orthogonal complement of the row space of \( A \) is the null space of \( A \). The orthogonal complement of the column space of \( A \) is the null space of \( A^T \) (sometimes called the left null space). Mathematically, this can be written:

\[
(row(A))^\perp = \text{null}(A) \text{ and } (\text{col}(A))^\perp = \text{null}(A^T)
\]

These four subspaces are called the fundamental subspaces of the matrix \( A \).
EXAMPLE

Let’s find bases for the four fundamental subspaces of the matrix \( A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \).

Suppose we know that \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) and \( \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Write down the dimensions of each fundamental subspace and describe the subspace-orthogonal complement pairs.

DEFINITION

Let \( W \) be a subspace of \( \mathbb{R}^n \) and let \( \{ \vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n \} \) be an orthogonal basis for \( W \). For any vector \( \vec{v} \) in \( \mathbb{R}^n \), the orthogonal project of \( \vec{v} \) onto \( W \) is defined as

\[
\text{proj}_W(\vec{v}) = \sum_{j=1}^{n} \text{proj}_{\vec{w}_j}(\vec{v}) = \sum_{j=1}^{n} \frac{\vec{v} \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \vec{w}_j
\]

The component of \( \vec{v} \) orthogonal to \( W \) is the vector \( \text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v}) \)

NOTE: this implies that \( \vec{v} = \text{perp}_W(\vec{v}) + \text{proj}_W(\vec{v}) \) (Draw a picture in \( \mathbb{R}^2 \))
Theorem 5.11
Let \( W \) be a subspace of \( \mathbb{R}^n \) and let \( \vec{v} \) be ANY vector in \( \mathbb{R}^n \). THEN there exist unique vectors \( \vec{w} \) in \( W \) and \( \vec{w}^\perp \) in \( W^\perp \) such that \( \vec{v} = \vec{w} + \vec{w}^\perp \). This theorem is known as the **Orthogonal Decomposition Theorem**. Note: a corollary of this theorem is that \((W^\perp)^\perp = W\).

**EXAMPLE**
Consider the subspace \( W \), \( x - y + 2z = 0 \) with the vector \( \vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \). Show that the orthogonal decomposition of \( \vec{v} \) is \( \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} \) and \( \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix} \).

Theorem 5.13
Let \( W \) be a subspace of \( \mathbb{R}^n \) then \( \dim(W) + \dim(W^\perp) = n \).

A corollary of Theorem 5.13 becomes clear when one applies it to the associated subspaces of a \( m \times n \) matrix \( A \). This is known as **The Rank Theorem**.
\( \dim(\text{row}(A)) + \dim(\text{null}(A)) = n \) and \( \dim(\text{col}(A)) + \dim(\text{null}(A^T)) = m \)

**The Rank Theorem**
If \( A \) is an \( m \times n \) matrix, then \( \text{rank}(A) + \text{nullity}(A) = n \) and \( \text{rank}(A) + \text{nullity}(A^T) = m \).
(Recall, \( \text{rank}(A) = \text{rank}(A^T) \))