Class 26: Friday April 7

**TITLE** Orthogonality and Projections Revisited

**CURRENT READING** Poole 5.1

**Summary**
We shall return to the investigation of projections and orthogonality, this time with more increased generality.

**Homework Assignment**
Poole, Section 5.1: 3, 4, 5, 6, 8, 9, 12, 13, 16, 17, 30, 31. EXTRA CREDIT 28, 33.

1. **Orthogonal Bases**

**DEFINITION**
An **orthogonal basis** of a subspace \( W \) of \( \mathbb{R}^n \) is a basis of \( W \) that is an **orthogonal set** of vectors. An orthogonal set of vectors is a collection of vectors \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k \} \) where *every* pair of distinct vectors is orthogonal to each other, i.e. \( \vec{v}_i \cdot \vec{v}_j = 0 \) for all \( i \neq j \).

**Theorem 5.1**
If \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k \} \) is an orthogonal set of nonzero vectors in \( \mathbb{R}^n \) then those vectors are linearly independent.

**EXAMPLE**
Show that \[
\begin{pmatrix}
2 \\
1 \\
-1
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
\] and \[
\begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix}
\] form an orthogonal basis for \( \mathbb{R}^3 \).

**Theorem 5.2**
Let \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k \} \) be an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \) and let \( \vec{w} \) be any vector in \( W \). THEN the unique scalars \( c_1, c_2, c_3, \ldots, c_n \) (also known as coordinates) where \( \vec{w} = \sum_{i=1}^{n} c_i \vec{v}_i \) are given by

\[
c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}
\]

**EXAMPLE**
Let’s show how this formula for the coordinates is derived. (Doesn’t it look familiar??)
Exercise

Given the orthogonal basis \( \beta = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \) and the vector \( \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) find the coordinates of \( \vec{w} \) with respect to \( \beta \), i.e. \([\vec{w}]_\beta\).

DEFINITION
An orthonormal basis of a subspace \( W \) of \( \mathbb{R}^n \) is a basis of \( W \) that consists of an orthonormal set of vectors. An orthonormal set of vectors is a collection of orthogonal unit vectors \( \{\vec{q}_1, \vec{q}_2, \vec{q}_3, \ldots, \vec{q}_k\} \) where \( \vec{q}_i \cdot \vec{q}_j = \delta_{i,j} \). The symbol \( \delta_{i,j} \) is known as the Kronecker delta function and has the property that \( \delta_{i,j} = 0 \) when \( i \neq j \) and \( \delta_{i,j} = 1 \) when \( i = j \).

Exercise

Form an orthonormal basis for \( \mathbb{R}^3 \) from the orthogonal basis \( \beta \) given in the previous Exercise.

2. Orthogonal Matrices

DEFINITION
A \( n \times n \) matrix \( Q \) is said to be an orthogonal matrix if the columns (and rows) of the matrix form an orthonormal set.

Theorem 5.4
The columns of an \( m \times n \) matrix \( Q \) form an orthonormal set if and only if \( Q^T Q = I_n \).

Theorem 5.5
A square matrix \( Q \) is orthogonal if and only if \( Q^{-1} = Q^T \).

Theorem 5.8
Let \( Q \) be an orthogonal matrix.

(a) \( Q^{-1} \) is orthogonal.
(b) \( \det(Q) = \pm 1 \).
(c) If \( \lambda \) is an eigenvalue of \( Q \), then \( |\lambda| = 1 \).
(d) If \( Q_1 \) and \( Q_2 \) are orthogonal \( n \times n \) matrices, then so is \( Q_1 Q_2 \).

EXAMPLE
Let’s form a square orthogonal matrix from the orthonormal basis found in the previous exercise and illustrate some of the results from Theorem 5.4, 5.5 and 5.8.