Class 21: Friday March 24

**Title**  Determinants  
**Current Reading**  Poole 4.2

**Summary**
Let’s understand how to compute the determinant of a $n \times n$ matrix and understand the properties and applications of determinants.

**Homework Assignment**
Poole, Section 4.2: 4, 6, 7, 10, 15, 26, 27, 32, 33, 48, 49, 50, 51, 52. EXTRA CREDIT 46.

**Definition**
Let $A$ be any matrix. The $ij$-minor of $A$ is the matrix obtained by removing its $i$th row and its $j$th column. It is denoted by $\hat{A}_{ij}$.

**Theorem 4.1**
Let $A$ be any $n \times n$ matrix. The determinant of $A$ is defined as:
If $n = 1$, then $|A| = A_{11}$.
If $n \geq 2$, then

$$|A| = A_{11}|\hat{A}_{11}| - A_{12}|\hat{A}_{12}| + \cdots + A_{1n}|\hat{A}_{1n}|$$

More general definition: Fix any row $i$. Then,

$$|A| = \sum_{j=1}^{n} A_{ij}(-1)^{i+j}|\hat{A}_{ij}| = \sum_{j=1}^{n} A_{ij}C_{ij}$$

Or, fix any column $j$. Then,

$$|A| = \sum_{i=1}^{n} A_{ij}(-1)^{i+j}|\hat{A}_{ij}| = \sum_{i=1}^{n} A_{ij}C_{ij}$$

Although it may seem like it, this definition is not really a “circular” definition. It’s known as a recursive definition. The formula is called the Laplace Expansion Theorem. The number $C_{ij} = (-1)^{i+j}|\hat{A}_{ij}|$ is known as the $i, j$-cofactor of $A$.

**Example**
Let’s compute the determinant of

$$\begin{vmatrix} 1 & 4 & 2 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

Ans: $1(10 - 1) - 4(6 - 0) + 2(3 - 0) = -9$

**Exercise**
Compute the determinant of the matrix using a different row and column.
1. **Properties of the Determinant of a Matrix**

Here is a summary of the various properties of the determinant.

1. The determinant is a linear function of the first row.

2. The determinant changes sign when two rows are exchanged.

3. The determinant of the $n$ by $n$ identity matrix is 1.

4. If two rows of $A$ are equal, then $\det A = 0$.

5. Subtracting a multiple of one row from another row leaves $\det A$ unchanged.

6. A matrix with a row of zeros has $\det A = 0$.

7. If $A$ is a triangular matrix the $\det A$ equals the product of the main diagonal entries.

8. If $A$ is singular then $\det A = 0$. If $A$ is invertible then $\det A \neq 0$.

9. The determinant of $AB$ is the product of the separate determinants: $|AB| = |A||B|$.

10. The transpose $A^T$ has the same determinant as $A$.

[GroupWork]

Come up with examples to illustrate each of the ten principles above.
2. Applications of the Determinant

**Theorem 4.7**
If $A$ is an $n$ by $n$ matrix then $|kA| = k^n|A|$

**Theorem 4.9**
If $A$ is an invertible $n$ by $n$ matrix then $|A^{-1}| = rac{1}{|A|}$

**EXAMPLE**
The sum of the determinant of two matrices is **NOT** equal to the determinant of the sum of two matrices, i.e. $|A + B| \neq |A| + |B|$. Let's prove this.

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**Cramer’s Rule**

Suppose we’re given a linear system of equations $A\vec{x} = \vec{b}$, where $A$ and $\vec{b}$ are given, and we are to find $\vec{x}$. We have learned how to solve this using Gaussian Elimination. A longer way to find $\vec{x}$ is as follows!

**Theorem** (Cramer’s Rule) Given $A\vec{x} = \vec{b}$, the $j$th coordinate of $\vec{x}$ is given by the formula

$$x_j = \frac{\det(B_j)}{\det(A)}$$

where $B_j$ is obtained by replacing the $j$-th column of $A$ by $\vec{b}$.

**Example 1.** Solve the system $2x + 4y = 1$ $x + 3y = 2$ using Cramer’s Rule.

Ans:

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Cramer’s Rule takes a lot more work than Gaussian Elimination to solve a system. So why is it useful? I think because it gives us a formula for the solution, as opposed to Elimination, which is only a procedure for finding the solution.
**Formula for \( A^{-1} \)**

**Theorem**  If \( A \) is an invertible matrix, then \( A^{-1} \) is given by

\[
(A^{-1})_{i,j} = \frac{C_{j,i}}{\det(A)}
\]

where \( C_{j,i} \) is the cofactor of \( A_{j,i} \).

Find the inverse of
\[
\begin{bmatrix}
2 & 6 & 2 \\
0 & 4 & 2 \\
5 & 9 & 0
\end{bmatrix}
\]
using the co-factor formula

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**The vector cross product**

**Definition 1.** Let \( \vec{u} \) and \( \vec{v} \) be vectors in \( \mathbb{R}^3 \). Their **cross product** is defined as

\[
\vec{u} \times \vec{v} = \det \begin{bmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{bmatrix}.
\]

**Note.** \( \hat{i} = (1,0,0) \), \( \hat{j} = (0,1,0) \), and \( \hat{k} = (0,0,1) \). These are unit vectors. But in computing the above determinant, just treat them as symbols or numbers. **Note:** the result of a cross-product is a vector which is orthogonal (perpendicular) to both vectors in the product. (Where could this be useful?)

Use the cross-product to find the general equation \( Ax + By + Cz = D \) of the plane in \( \mathbb{R}^3 \) which is the span
\[
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \begin{bmatrix}
-2 \\
1 \\
3
\end{bmatrix}
\]