Class 19: Monday March 20

TITLE Application of Linear Algebra: Graph Theory

CURRENT READING Poole 3.6

Summary
Let’s explore an interesting application of matrices to graph theory.

Homework Assignment
Poole, Section 3.7: 27, 28, 29, 30, 35, 36, 37, 38.

DEFINITION
A graph consists of a finite set of points called vertices and a finite set of edges which connect two (not necessarily distinct) vertices. Two vertices are said to be adjacent if they are the endpoints of an edge.

If $G$ is a graph with $n$ vertices, then its adjacency matrix is the $n \times n$ matrix $A$ (sometimes written $A(G)$) defined by:

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

EXAMPLE
Consider the adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Draw the associated graph.

DEFINITION
A path in a graph is defined to be a sequence of edges that allows travel from one vertex to another continuously. The length of a path is the number of edges the path contains. A path with $k$ edges is called a $k$-path. When a path begins and ends with the same vertex it is called closed and the path is called a circuit. A path that does NOT include the same edge more than one is called a simple path.

Exercise
Show that $A^2 = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Given $A$ is the adjacency matrix of a graph $G$, the $(i, j)$ entry in the matrix $A^k$ is equal to the number of $k$-paths between vertices $i$ and $j$.

EXAMPLE
Show that there are SIX 3-paths between the first vertex and second vertex.
DEFINITION
A graph with directed edges is known as a **digraph**. If $G$ is a digraph with $n$ vertices, then its **adjacency matrix** is the $n \times n$ matrix $A$ (sometimes written $A(G)$) is defined by:

$$a_{ij} = \begin{cases} 
1 & \text{if there is an edge FROM vertices } i \text{ TO } j \\
0 & \text{otherwise}
\end{cases}$$

**Exercise**

Given $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \end{bmatrix}$, draw the corresponding directed graph below.

**GROUP WORK**

Consider the following 5-player **tournament** (a directed graph with exactly one directed edge between every pair of vertices) representing results between 5 tennis players Davenport, Graf, Hingis, Seles and (Venus) Williams. A directed edge from vertex $i$ to vertex $j$ means that Player $i$ defeated Player $j$.

The results are as follows:
- Davenport defeated Graf, Seles, and Williams, lost to Hingis.
- Graf defeated Hingis, Seles, and Williams, lost to Davenport.
- Hingis defeated Davenport and Seles, lost to Graf and Williams.
- Seles defeated Williams, lost to Davenport, Graf and Hingis.
- Williams defeated Hingis, lost to Davenport, Graf and Seles.

(a) Draw a picture of the digraph representing these results.

(b) Write down the adjacency matrix for the digraph you drew. (What must all the entries of the main diagonal be?)

(c) Rank the players. **No Ties Allowed.** What matrix operations do you have to do comeup with a reasonable ranking order? **DISCUSS YOUR STRATEGIES.**