SUMMARY Matrix Properties
CURRENT READING Poole 3.1

Summary
We begin our study of Chapter 3 by considering matrices as objects in their own right, and not just as ways of viewing, or parts of, linear systems.

Homework Assignment
HW # 10: Section 3.1: 1,2,3,4,5,6,7,8,34,35. EXTRA CREDIT 37: DUE WED FEB 15

1. Matrix Definitions

DEFINITION
Let $A$ be an $m \times n$ matrix (with $m$ rows and $n$ columns). If $m = n$, then $A$ is said to be a square matrix. For $1 \leq i \leq m$ and $1 \leq j \leq n$, the $(i,j)$-entry of $A$, denoted by $A_{i,j}$, is the number in the $i$th row and the $j$th column of $A$. We denote the $i$th row of $A$ by $\text{row}_i(A)$, and the $j$th column of $A$ by $\text{col}_j(A)$.

Note. For convenience, some books, including ours, drop the comma from $A_{i,j}$, and instead write $A_{ij}$. You may do this too, except when it can cause ambiguity, as in: $A_{123} = A_{12,3}$ or $A_{1,23}$?

Q: An $m$-component column vector is a ?×? matrix? A:

Q: An $n$-component row vector is a ?×? matrix? A:

DEFINITION
Let $A$ and $B$ be $m \times n$ matrices. Then their sum $A+B$ is an $m \times n$ matrix $C$ defined by: $C_{i,j} = A_{i,j} + B_{i,j}$.

Example 1. Compute $B + A$, where $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

DEFINITION
Let $A$ and $B$ be $m \times n$ matrices. Then $A$ is said to be equal to $B$ if both $A$ and $B$ have the same dimensions and if $A_{i,j} = B_{i,j}$.

Example 2. Q: Are $\begin{bmatrix} 1 & 0 & -3 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ equal? A: No! (Why not?)

DEFINITION
Let $A$ be a $m \times n$ matrix and $c$ is a real number. Then $cA$ is said to a scalar multiple of $A$ and $cA$ is obtained by multiplying each element of $A$ by $c$. 


DEFINITION
Let $O$ be a $m \times n$ matrix called the **zero matrix** where every entry equals zero. Clearly, $A + O = O + A = A$ and $A - A = -A + A = O$. The zero matrix acts like the matrix “additive identity” also known as the number “zero.”

2. Matrix Multiplication
We add matrices component-wise: $(A+B)_{i,j} = A_{i,j} + B_{i,j}$. But we do not multiply matrices component-wise: $(AB)_{i,j} \neq A_{i,j}B_{i,j}$ (just as vector addition is component-wise, but the dot product isn’t).

DEFINITION
Let $A$ be an $m \times n$ matrix, and $B$ an $n \times q$ matrix. Then their **product** $AB$ is an $m \times q$ matrix $C$ defined by $C_{i,j} = \text{row}_i(A) \cdot \text{col}_j(B)$. (Equivalently, $C$ can be defined by: $\text{col}_j(C) = A \text{col}_j(B)$.)

**Example 3.** Compute $BA$, where $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

**Q:** What type of matrix can be multiplied by itself? **Ans:** A square matrix.

**Notation:** $AA = A^2$, $AAA = A^3$, $\cdots$. Also, note that $A^rA^s = A^{r+s}$ and $(A^r)^s = A^{rs}$ when $r$ and $s$ are non-negative integers.

**Example 4.** Compute $A^2$ and $A^3$

DEFINITION
The $n \times n$ **identity matrix** $I$ or $I_n$ is a square matrix defined to have 1’s along its diagonal, and 0’s elsewhere. The identity matrix acts like the matrix “multiplicative identity” also known as the number “one.” Clearly, $AI = IA = A$.

DEFINITION
Two $n \times n$ matrices $A$ and $B$ are said to be **inverses** of each other if $AB = I_n$ and $BA = I_n$.

3. Matrix Transposes

DEFINITION
Given a matrix $A$, the transpose matrix is denoted $A^T$. The rows of $A$ become the columns of $A^T$. If $A$ is $m \times n$ then $A^T$ is $n \times m$. Specifically, $A^T_{ij} = A_{ji}$.
4. Properties of the Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(cA)^T = cA^T$
- $(A^r)^T = (A^T)^r$ for non-negative integers $r$

- Recall that $A\vec{x}$ is a linear combination of the columns of $A$, so $x^T A^T$ is a linear combination of the ROWS of $A^T$

**Exercise**

Confirm the above transpose properties with $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

5. Symmetric Matrices

**Definition**

A matrix is said to be *symmetric* if it is its own transpose, i.e. $A^T = A$.

The inverse of a symmetric matrix is also symmetric.

The product of a matrix with its transpose produces a symmetric matrix.

$R^T R = R^T (R^T)^T = R^T R$
6. Block Matrices

Consider \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) and \( B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} \)

**Exercise**

Write down \( AB \) in terms of the elements of \( A \) and \( B \).

Now, suppose the elements of \( A \) and \( B \) are themselves matrices!

\[
A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 4 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \\ 7 & 2 \end{bmatrix}
\]

\[
B_{11} = \begin{bmatrix} 4 & 3 \\ -1 & 2 \\ 1 & -5 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}, \quad B_{13} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad B_{23} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

Now compute \( AB \) block by block. First check that matrix \( A \) and \( B \) are **partitioned conformably for block multiplication**. (In other words, that in every possibly matrix multiplication the dimensions match up properly.)