Linear Systems

Class 8: **Wednesday February 8**

SUMMARY Linear Independence and Span
CURRENT READING Poole 2.3

Summary
We will discuss the concepts of linear independence, linear dependence and spanning.

Homework Assignment
HW #7: Section 2.2: 1, 2, 3, 4, 5, 6, 7, 8, **22, 23,26**, *27, 28, 29, 36,* **43**. *EXTRA CREDIT # 47.*

RECALL: A homogeneous linear system ((i.e. one where the right hand side or constant term in each equation is equal to zero) always has at least one solution, so it is ALWAYS a **consistent** system.

Theorem
A homogeneous system has infinitely many non-zero solutions if it has more variables than equations (i.e. $n > m$).

In other words, there are always free variables when the number of variables (n) is greater than the number of equations (m) in a linear system.

GroupWork
Can you think of a **geometric** or visual representation of this fact?

Linear independence

Let's revisit the question of when do we know that a linear combination of a set of vectors equals a given vector.

Previously you had been told that **every vector** in $\mathbb{R}^2$ can be expressed as a linear combination of $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

How can we show this? How about if we denote "any vector in $\mathbb{R}^2$" to be $\begin{bmatrix} a \\ b \end{bmatrix}$ and attempt to solve the vector equation: $x \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

where x and y are unknown scalars that we will try to determine **assuming** a and b exist and putting no conditions on a and b since they can be any real number, so that (a, b) is any point in the plane $\mathbb{R}^2$.

EXAMPLE
Let's form the augmented matrix $[A|\vec{b}]$ where $A = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} a \\ b \end{bmatrix}$, and then apply row reduction.

$$\begin{bmatrix} 3 & 1 & | & a \\ 1 & 5 & | & b \end{bmatrix} \rightarrow$$

Exercise

Show that the solution of the previous question is that $x = \dfrac{5a-b}{14}$, $\quad y = \dfrac{3b-a}{14}$ so that

$$\frac{5a-b}{14}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3b-a}{14}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ for ANY } a \text{ and } b$$

Theorem 2.4

The above result can be generalized into a theorem, which is that: A system of linear equations with augmented matrix $[A|\vec{b}]$ is **consistent** if and only if $\vec{b}$ is a linear combination of the columns of A.

Discussion

Q: What does "if and only if" mean?
A: It means that the logical implication "goes both ways." In other words, if the statement after "**if and only if**" is true, then it implies the statement BEFORE it is true, **AND** if the statement before the "**if and only if**" is true then that implies the statement after it is true.

Exercise

Is $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ a consistent system?

What does the set of all possible linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ look like?

Definition: span

The **span** of a set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}$ is the set of all linear combinations of those vectors.

$$\text{span}(\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}) = \{c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} \mid c_i \in \mathbb{R}\}$$

A set S of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ in **spans** a vector space if their linear combinations fill the space. (That is, every vector in the space can be written a s alinear combination of vectors from the set.) The set $S = \text{span}(\vec{v_1}, \vec{v_2}, \ldots \vec{v_n})$ is called a **spanning set** for the vector space for the vector space.

NOTE

"Span" can be used both as a noun and as a verb. Typically, the vector spaces we are talking about are $\mathbb{R}^n$ but this definition applies to more exotically defined vectors and vector spaces.

Example 1. **Q:** Is (2,3) in the span of $v_1 = (0, 1)$ and $v_2 = (1, 0)$? **A:** Yes. why?

Q: Is $(2, 3) \in \text{span}\{(1, 1), (2, 2)\})$? **A:** No. Why?

Q: Is $(2, 3) \in \text{span}\{(1, 1), (1, 0), (0, 1)\}$? **A:** Yes. Why?

Q: Do $(1, 1)$ and $(2, 2)$ span $\mathbb{R}^2$? **A:** No. Why?

Q: Do $(1, 0)$ and $(0, 1)$ span $\mathbb{R}^2$? **A:** Yes. Why?

Q: Describe the span of $\{(1, 3)\}$. **A:** The line $y = 3x$ in the xy-plane.

Exercise

What is the span of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$?

Definition: linear independence

A set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ is **linearly independent** provided

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$$

if and only if $c_i = 0$ for $i = 1, 2, \ldots, n$. (The **only** way to combine linearly independent vectors to get the zero vector is to multiply them all by zero scalars.)

Definition: linear dependence

A set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ is **linearly dependent** provided one of the vectors is a linear combination of the others. (So there is a way to combine **linearly dependent** vectors to get the zero vector by using non-zero scalars.)

Example 2. Let $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Q: Are $\vec{v_1}, \cdots, \vec{v_4}$ linearly independent? **A:** No. Why?

Q: How about $\vec{v_1}, \vec{v_2}, \vec{v_3}$? **A:** Yes. Why?

Q: Describe $W = \text{span}(\vec{v_1}, \cdots, \vec{v_4})$.

NOTE: The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent and span W.

Example 3. Are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ linearly independent? What about $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$? **Explain.**