Class 6: Friday February 3

SUMMARY  Solving Linear Systems
CURRENT READING  Poole 2.1

OUTLINE
Now that we can visualize and understand the basic nature of linear systems, let’s learn some direct techniques for finding solutions of a given linear system.

Homework Assignment
HW #6: Section 2.1 # 1, 3, 10, 11, 13, 15, 17, 20, 23, 28, 29, 32, 34, 35, 44: DUE MON FEB 6

Warm-Up
Q: How many solutions is it possible for a linear system to have?
A:

1. Elimination

Consider the following linear system of 2 equations in 2 unknowns:

\[
\begin{align*}
x + y &= 4 \\
2x - 3y &= 1 
\end{align*}
\]

Using algebra one can transform this system into an EQUIVALENT form from which the solution can be easily found by back-substitution. This transformation process is called elimination.

\[
\begin{align*}
x + y &= 4 \\
-5y &= -7 
\end{align*}
\]

Using elimination one tries to change the coefficient matrix from \( \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \) to \( \begin{bmatrix} 1 & 1 \\ 0 & -5 \end{bmatrix} \). The transformed coefficient matrix is said to now be upper triangular.

On the axes below show how the graphical representations of the equivalent systems change, but the SOLUTION of the system remains the same (obviously).
2. Equivalent Systems of Equations

**DEFINITION: solution**
Suppose we have a system of equations in \( n \) variables, \( x_1, \ldots, x_n \). An \( n \)-component vector \((c_1, \ldots, c_n)\) is said to be a **solution** for the system if substituting \( c_i \) for \( x_i \) (for all \( i = 1, \ldots, n \)) simultaneously satisfies all the equations.

**DEFINITION: equivalent system**
Two systems of linear equations are said to be **equivalent** if they have the same solutions (i.e., if any solution of one system is also a solution of the other).

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**Example 1.** Q: Are the following systems equivalent? Why or why not?

\[
\begin{align*}
x + 2y &= 4 \\
3x - y &= 5
\end{align*}
\]

vs.

\[
\begin{align*}
4x + y &= 9
\end{align*}
\]

(eq1 + eq2 in system1 gives the eq in system2)

Ans: __________ Why?

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**Example 2.** Q: How about the following two systems, are they equivalent?

\[
\begin{align*}
x + 2y &= 4 \\
3x - y &= 5
\end{align*}
\]

vs.

\[
\begin{align*}
x + 2y &= 4 \\
6x - 2y &= 10
\end{align*}
\]

Ans: ________________ Why?

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**Free variables**

**Example 3.** Solve the following system:

\[
\begin{align*}
x + y + z &= 1 \\
x + 2y + z &= 3
\end{align*}
\]

Q: How many solutions does this system have? **Ans.** ________________

Write down the solution using \( z \) as a free variable.

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3. Standardizing The Elimination Process

**DEFINITION: row operation**

An **elementary row operation** is any of the following, where \( c \) is a nonzero scalar:

1. \( \text{row}_i = \text{row}_i + c(\text{row}_k) \);
2. \( \text{row}_i = c(\text{row}_i) \);
3. \( \text{row}_i = \text{row}_k \), and \( \text{row}_k = \text{row}_i \) (switch rows).

**EXAMPLE**

Let’s write down a example of applying each of the elementary row operations to the matrix

\[
\begin{bmatrix}
4 & 6 & 3 \\
-7 & -1 & 8 \\
2 & 0 & -1
\end{bmatrix}
\]
DEFINITION: pivot
A pivot is the first non-zero entry in a row.

DEFINITION: row reduction
The process of applying elementary row operations to a matrix to eliminate coefficients (turn them into zero) is called row reduction.

ALGORITHM: Gaussian elimination
1. Find leftmost pivot.
2. If necessary, do row-exchange to "bring up the pivot."
3. (Optional) Divide to make pivot = 1.
4. Make zeros under pivot.
5. Find next leftmost pivot.
6. Go to step 2.

Consider the system
\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & -1 & 1 \\
3 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
=
\begin{bmatrix}
9 \\
8 \\
3
\end{bmatrix}
\]
which corresponds to the linear system of equations
\[
\begin{align*}
1x + 2y + 3z &= 9 \\
2x - 1y + 1z &= 8 \\
3x - 1z &= 3
\end{align*}
\]

We want to get the system into a form which we can solve using back-substitution. To do this we need to identify pivots, multipliers and look to see if any row exchanges will be necessary.

In this system, the coefficient of \(x\) in the first row is the pivot. We multiply the pivot by the coefficient of \(x\) in the second row and subtract rows. The multiplier is the number we have to multiply the pivot by to eliminate the coefficient we want. This eliminates \(x\) in the second row.

In our example above the pivot is \_______\. The first multiplier is \_______\.

We repeat the process to eliminate the coefficient of \(x\) in the third row. The result is that the first column ends up with zeroes beneath the pivot. **NOTE: We include the right-hand side in our calculations by forming an AUGMENTED COEFFICIENT MATRIX**

\[
\begin{bmatrix}
1 & 2 & 3 & | & 9 \\
2 & -1 & 1 & | & 8 \\
3 & 0 & -1 & | & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & | & 9 \\
0 & -5 & -5 & | & -10 \\
0 & -6 & -10 & | & -24
\end{bmatrix}
\]

Now we proceed to the second unknown variable \(y\) and again look for pivots and multipliers.

The pivot in the second row is \_______\ and the multiplier is \_______\.

If we continue the process of elimination:

\[
\begin{bmatrix}
1 & 2 & 3 & | & 9 \\
0 & -5 & -5 & | & -10 \\
0 & -6 & -10 & | & -24
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & | & 9 \\
0 & -5 & -5 & | & -10 \\
0 & 0 & -4 & | & -12
\end{bmatrix}
\]

We have now achieved upper-triangular form so we can solve the system by back-substitution.
You should find that \[
\begin{bmatrix}
2 \\
-1 \\
3
\end{bmatrix}
\] is the exact solution to \[
\begin{bmatrix}
1 & 2 & 3 \\
2 & -1 & 1 \\
3 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
9 \\
8 \\
3
\end{bmatrix}
\]

Congratulations on your first application of the Gaussian Elimination algorithm!

4. Examples of elimination

Exercise
Consider the linear systems. Get them into upper-triangular form. Solve by back-substitution.

\[
\begin{align*}
1x + 1y + 1z &= 2 \\
2x - 2y + 6z &= 7 \\
1x - 1y + 3z &= 3
\end{align*}
\]

\[
\begin{align*}
0x + 4y &= 2 \\
1x - 3y &= 1
\end{align*}
\]

\[
\begin{align*}
1x - 1y + 3z &= 3 \\
2x + 1y + 1z &= 4 \\
1x - 1y - 1z &= -1
\end{align*}
\]