Test 1: Linear Systems

Math 214
Ron Buckmire

Friday March 3 2006
2:30pm-3:25pm

Name: ___________________________

Key

Directions: Read all problems first before answering any of them. There are 6 pages in this test. This is a one hour, no-notes, closed book, test. No calculators. You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your “scratch work.”

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1. Span, Linear Independence, Rank. 20 points.

Consider the matrix
\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\]

(a) (4 points.) Show that \(\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\).

\[
\begin{align*}
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} & \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\
& \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \\
& \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

(b) (4 points.) Given your knowledge of \(\text{rref}(A)\), what is the rank of the matrix \(A\)? EXPLAIN YOUR ANSWER.

\[\text{Rank is 3 (number of non-zero rows of matrix A)}\]

(c) (4 points.) Given your knowledge of \(\text{rref}(A)\), what is the span of the columns of matrix \(A\)? EXPLAIN YOUR ANSWER.

\[\text{Span of the columns } = \mathbb{R}^3 \text{ since the columns must be linearly independent since the rank = number of columns and no free variables}\]

(d) (4 points.) Given your knowledge of \(\text{rref}(A)\), discuss the linear independence of the columns of the matrix \(A\). EXPLAIN YOUR ANSWER.

\[\text{Columns (and rows) are linearly independent}\]

(e) (4 points.) Given your knowledge of \(\text{rref}(A)\), discuss whether the matrix \(A^{-1}\) exists. EXPLAIN YOUR ANSWER.

\[A^{-1} \text{ exists since } \text{rref}(A) = I.\]
2. Dot product, magnitude, lengths. 30 points.

Suppose the dot product $\mathbf{u} \cdot \mathbf{v}$ is re-defined to be just the product of the lengths of the vectors $\mathbf{u}$ and $\mathbf{v}$. Let's call this new dot product the Buckmire product and denote it

$$\mathbf{u} \circ \mathbf{v} = |\mathbf{u}||\mathbf{v}|$$

Discuss which of the following statements are true for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$ under the Buckmire product.

(a) (6 points.) $\mathbf{u} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{u}$  \textbf{TRUE}

*Multiplication is commutative.*

This statement is true for dot product \textbf{AND} the Buckmire product.

(b) (6 points.) $(c\mathbf{u}) \circ \mathbf{v} = c(\mathbf{u} \circ \mathbf{v})$  \textbf{FALSE}

$$(c\mathbf{u}) \circ \mathbf{v} = |c\mathbf{u}||\mathbf{v}|$$

$$= c|\mathbf{u}||\mathbf{v}| \neq c|\mathbf{u}||\mathbf{v}| \text{ if } c < 0$$

This statement is true for dot product, \textbf{FALSE} for Buckmire product.

(c) (6 points.) $\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = \mathbf{u} \circ \mathbf{v} + \mathbf{u} \circ \mathbf{w}$  \textbf{FALSE}

$$\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = |\mathbf{u}||\mathbf{v} + \mathbf{w}|$$

$$|\mathbf{v} + \mathbf{w}| \neq |\mathbf{v}| + |\mathbf{w}| \text{ for all } \mathbf{v}, \mathbf{w}$$

True for dot product, \textbf{not true} for Buckmire product.

(d) (6 points.) $\mathbf{u} \circ \mathbf{u} \geq 0$  \textbf{TRUE}

$$\mathbf{u} \circ \mathbf{u} = |\mathbf{u}||\mathbf{u}| = |\mathbf{u}|^2 \geq 0$$

This is \textbf{TRUE} for Buckmire and dot product.

(e) (6 points.) $\mathbf{u} \circ \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$.  \textbf{TRUE}

*If $\mathbf{u} = \mathbf{0} \Rightarrow |\mathbf{u}| = 0 \Rightarrow \mathbf{u} \circ \mathbf{u} = |\mathbf{u}||\mathbf{u}| = 0$, $\mathbf{0} = \mathbf{0}$

*If $|\mathbf{u}|^2 = 0 \Rightarrow |\mathbf{u}| = 0 \Rightarrow \mathbf{u} = \mathbf{0}$

\textbf{TRUE} for Buckmire product, \textbf{not TRUE} for dot product also.

\textbf{3}
3. Matrix Operations. 20 points.

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

Recall the zero matrix $\mathbf{0}$ and identity matrix $\mathbf{I}$ have particular properties in matrix arithmetic which often (but not always!) correspond to the properties the number zero and the number one that you know and love.

**NOTE:** $A$ is assumed to be a generic (unknown) $m \times n$ matrix for every part below.

(a) 5 points. TRUE or FALSE? “If $A^2 = \mathbf{0}$ then $A = \mathbf{0}$.”

**FALSE**

If $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{0}$

There exists an $A \neq \mathbf{0}$ for which $A^2 = \mathbf{0}$

(b) 5 points. TRUE or FALSE? “If $A = \mathbf{0}$ then $A^2 = \mathbf{0}$.”

**FALSE**

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$A^2$ is not defined.

If $A$ is square, then $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A^2 = A \neq \mathbf{0}$

(c) 5 points. TRUE or FALSE? “If $A^2 = \mathbf{I}$ then $A = \mathbf{I}$.”

**FALSE**

If $A = A^{-1}$ then $A \cdot A^{-1} = A^2 = \mathbf{I}$

A matrix whose inverse equals itself is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(d) 5 points. TRUE or FALSE? “If $A = \mathbf{I}$ then $A^2 = \mathbf{I}$.”

**TRUE**

$A = \mathbf{I} \Rightarrow AA = 2 \cdot 2 = 2$
4. Parametric equations, lines, planes, subspaces. 30 points.

Consider the object described parametrically by \( x = t + 1, y = 2t - 3, z = 3t \) in \( \mathbb{R}^3 \).

(a) (10 points.) Write down a system of three linear equations which has this object as its solution.

\[
\begin{align*}
    x &= 1 + t \quad \Rightarrow \quad t = x - 1 \\
    y &= 2t - 3 \quad \Rightarrow \quad t = \frac{y + 3}{2} \\
    z &= 3t \\

    \frac{x - 1}{2} &= \frac{y + 3}{2} \\
    \frac{2x - 2}{5} &= \frac{y + 3}{2} \\
    -3x + z &= -3 \\
    2z &= 3y + 9 \\
\end{align*}
\]

(b) (10 points.) What is the dimension of this object? What is the geometric interpretation of this solution to your linear system in (a)? Write down a vector equation describing this object.

\[
\begin{pmatrix}
    2 & -1 & 0 \\
    -3 & 0 & 1 \\
    0 & -3 & 2
\end{pmatrix}
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
=
\begin{pmatrix}
    5 \\
    -3 \\
    9
\end{pmatrix}
\]

This is the intersection of three planes in \( \mathbb{R}^3 \) along a line not through the origin.

\[
\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ \frac{1}{2} \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

This is a 1-D object (a line).

\[
\begin{pmatrix}
    2 & -1 & 0 & \frac{5}{2} \\
    -3 & 0 & 1 & -3 \\
    0 & -3 & 2 & 9
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    2 & -1 & 0 & \frac{5}{2} \\
    -1 & 1 & 2 & 2 \\
    0 & -3 & 2 & 9
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
    1 & 1 & -1 & -2 \\
    0 & 3 & 2 & 9 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

(c) (10 points.) Is this object a subspace of \( \mathbb{R}^3 \)? Prove your answer!

If \( t = 0 \), \( \vec{x} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} \neq \vec{0} 
This object is NOT a subspace of \( \mathbb{R}^3 \)

since the zero vector is NOT on the line.

\[
\vec{c} = c\begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} \neq \text{on line}
\]

It is also NOT closed under scalar multiplication or vector addition.

\[
\vec{x} + \vec{y} = 5\left( \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \right) = 5\begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} \text{ is in same direction}
\]

but not on line!
BONUS QUESTION. Linear Independence, Dependence, Inverse. (10 points.)
If possible, write down five different 3 x 3 matrices each one which has one of the following properties:
(i) MATRIX A: The rows are linearly independent but the columns are linearly independent.
(ii) MATRIX B: The rows are linearly dependent but the columns are linearly independent.
(iii) MATRIX C: The rows and columns are linearly independent.
(iv) MATRIX D: The rows are linearly dependent but the columns are linearly dependent.
(v) MATRIX E: The transpose of the matrix equals the inverse of the matrix.
EXPLAIN YOUR ANSWER THOROUGHLY. EXTRA CREDIT POINTS ARE HARD TO GET.

(i) Use example from Question 1.

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \]

\[ \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{I} \]

so columns & rows are lin indep

rows are lin ind

(ii) \[ B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ C^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix} \]

(iii) \[ D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]

columns & rows are all the same so you have lin dep rows and cols

(iv) \[ E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ (or \ any \ permutation \ matrix) \]