# Linear Systems


 

## Section 3.5: Independence, Basis, and Dimension

**Definition.** A set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ is **linearly independent** provided

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$$

if and only if $c_i = 0$ for $i = 1, 2, \ldots, n$. (The only way to combine the vectors to get the zero vector is to multiply them all by zero scalars.)

**Definition.** The **span** of a set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ is the set of all linear combinations of those vectors.

$$\langle \vec{v_1}, \vec{v_2}, \ldots \vec{v_n} \rangle = \{c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} \mid c_i \in \mathbf{R}\}$$

**Definition.** A set of vectors **spans** a vector space if their linear combinations fill the space.
**Definition.** A **basis** for a vector space is a set of vectors that

1. are linearly independent, and
2. span the space.

So we're looking for the smallest set of vectors (linearly independent) that span the space.

Example:

The vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ form a basis for $\mathbf{R}^3$.

The vectors $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, -4)$ also form a basis for $\mathbf{R}^3$.

In fact, there are infinitely many bases for the same space.

**Definition.** The **dimension** of a vector space $V$ equals the number of vectors in a basis of $V$.
**Note 1:** When using Strang's method for finding the nullspace of a matrix $A$, the "specific solutions" form a basis for $\mathcal{N}(A)$. Thus we see that the dimension of $\mathcal{N}(A)$ is equal to the number of free variables, $n - r$.
**Note 2:** Recall that linear combinations of the pivot columns form the column space. Thus these columns form a basis for $\mathcal{C}(A)$. We see that the dimension of $\mathcal{C}(A)$ is $r$.