Worksheet 25

**TITLE** Gradient Fields and Path Independence

**CURRENT READING** McCallum, Section 18.3

**HW #12 (DUE Wednesday 11/19/14 5PM)**
McCallum, *Section 18.1*: 6, 11, 12, 13, 14, 22, 27.
McCallum, *Section 18.2*: 4, 5, 6, 7, 8, 20, 33.
McCallum, *Section 18.3*: 3, 4, 5, 6, 18, 21, 30.

**SUMMARY**
This worksheet discusses when one can use then Fundamental Theorem of Line Integrals to more simply evaluate line integrals regardless of the path taken.

**RECALL**
The Fundamental Theorem of Calculus says

\[ \int_{a}^{b} f'(t) \, dt = f(b) - f(a) \]

There is a corresponding principle for line integrals, called the Fundamental Theorem of Calculus for Line Integrals.

**The Fundamental Theorem of Line Integrals**

**THEOREM**
Given that \( C \) is a piecewise smooth oriented path which starts at \( \vec{x}_A \) and ends at \( \vec{x}_B \). If \( f \) is a function whose gradient \( \nabla f \) is continuous on the path \( C \), then

\[ \int_{C} \nabla f \cdot d\vec{r} = f(\vec{x}_B) - f(\vec{x}_A) \]

**CONCEPTUAL UNDERSTANDING**
The fundamental Theorem means that regardless of the path taken from \( \vec{x}_A \) to \( \vec{x}_B \) the value of a line integral in a gradient field is the same, in other words **line integrals of gradient fields are path-independent**. This is fantastic because it means we can evaluate line integrals without having to worry about parametrizations of paths whatsoever. In other words, **ALL GRADIENT FIELDS ARE PATH-INDEPENDENT**.

**DEFINITION: conservative or path-independent vector field**
A vector field \( \vec{F} \) is said to be **conservative** or **path-independent**, if for any two points \( \vec{x}_A \) and \( \vec{x}_B \), the line integral \( \int_{C} \vec{F} \cdot d\vec{r} \) has the same value along ANY piecewise smooth path \( C \) lying in the domain of \( \vec{F} \) that connects the points \( \vec{x}_A \) and \( \vec{x}_B \).
EXAMPLE

Given that the vector field \( \vec{F}(x,y) = \nabla f \) where \( f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} \) find the value of \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the circular arc in the counterclockwise direction from \((2,0)\) to \((0,2)\).

THEOREM

If a vector field \( \vec{F} \) is a continuous and path-independent on an open region \( \mathcal{R} \), then there exists an \( f \) defined on \( \mathcal{R} \) so that \( \text{grad} \ f \) equals \( \vec{F} \).

DEFINITION: potential function

This function \( f \) which can be used to generate a path-indendent field (i.e. a gradient field) is called the potential function for the vector field \( \vec{F} \). Physicists and applied mathematicians like to use the symbol \( \phi \) for the potential, so that \( \vec{F} = \nabla \phi \).

CONCEPTUAL UNDERSTANDING

The theorem says that all path-independent fields \( \vec{F} \) must possess a potential function \( f \) which can be used to generate the field \( \vec{F} \) by taking the gradient of \( f \). But this means that path-independent fields are gradient fields.

ALL PATH-INDEPENDENT VECTOR FIELDS ARE GRADIENT FIELDS

So, one way to show that a give vector field is path-independent is to find a potential function for that field.

Exercise

McCallum, page 978, Example 3. Show that the vector field \( \vec{F}(x,y) = y \cos xy \hat{i} + (\sin x + y) \hat{j} \) is path-independent.