Worksheet 22

**TITLE** The Calculus Of Curves In Space  
**CURRENT READING** McCallum, Section 17.1-17.2  
**HW #8 (DUE Wednesday 11/4/14 5PM)**  
McCallum, *Section 16.3*: 2, 5, 6, 28, 39, 40, 41, 42, 54* *, 55*.  
McCallum, *Chapter 16.4*: 3, 7, 8, 17, 20, 22.  
McCallum, *Chapter 16.5*: 12, 13, 14, 15, 21, 22, 23, 63*, 73.  
McCallum, *Chapter 16 Review*: 1, 4, 10, 11, 12, 14, 20, 23, 55*, 56*.  

**SUMMARY**  
This worksheet discusses curves in space, primarily parametric equations in $\mathbb{R}^3$ of curves in space. In addition, we will learn about the physical significance of derivatives of these vector functions of scalar variables, i.e. $\vec{x}(t)$.

**RECALL:** The equation of a line in space can be written as $\vec{x} = \vec{p} + \vec{d}t$ where $\vec{p}$ is the position vector for a point on the line and $\vec{d}$ is the direction (i.e. displacement) in which points on the line move.

**Parametric Equation Of A Line In $\mathbb{R}^3$**  
Suppose we have linear equations $x = a + bt$, $y = c + dt$, $z = e + ft$ where $a, b, c, d, e$ and $f$ are known numbers

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ c \\ e \end{bmatrix} + t \begin{bmatrix} b \\ d \\ f \end{bmatrix} = \vec{p} + \vec{d}t$$

**EXAMPLE**  
Which of the following set of parametric equations is a representation for a line in $\mathbb{R}^3$?

$$x = 1 - 2t, \quad y = t, \quad z = 2 + t^2$$

OR

$$x = 1 - 2t^2, \quad y = t^2 - 4, \quad z = 2 + t^2$$

**Exercise**  
Sketch the curve being described by $\vec{x} = \begin{bmatrix} 2t - 4 \\ t^2 + 3 \end{bmatrix}$ for all $t \in \mathbb{R}$
GROUP WORK
In each case sketch the curve being described by the following parametrizations

(a) \( x(t) = t^3, \quad y(t) = t^6 \) for \(-\infty < t < \infty\)
(b) \( x(t) = t^2, \quad y(t) = t^4 - 1 \) for \(0 \leq t \leq 2\)
(c) \( x(t) = \cos(t^2), \quad y(t) = 2\sin(t^2) \) for \(0 \leq t \leq \sqrt{\pi}\)
(d) \( x(t) = t^2, \quad y(t) = t \) for \(-2 \leq t \leq 2\)

Parametrized Curves

**DEFINITION:** parametrization of a curve in \( \mathbb{R}^2 \)

A path traced by a point \( P = (x, y) \) where \( x \) and \( y \) are functions of a parameter \( t \) is called a **parametric curve** or parametrized path. We say that \( \vec{x} = (x(t), y(t)) \) where \( \vec{x} \) is the position vector of the points on the curve.

**NOTE:**

We could expand these parametrization to \( \mathbb{R}^n \) if we had \( n \) equations in the parametric variable.

Parametrizations are not unique. Any path can be parametrized in an infinite number of different ways. For example, the curve \( y = f(x) \) can always be parametrized by \( x = t, y = f(t) \).

The path is different from the curve. The path is traversed in the direction of increasing values of the parameter, while the curve is simply the set of points the path follows.

In \( \mathbb{R}^2 \) the slope of the tangent line to the parametrized curve at any point \( \frac{dy}{dx} \) can be found by the ratio of \( \frac{dy}{dt} \) over \( \frac{dx}{dt} \) provided that \( x(t) \) and \( y(t) \) are differentiable functions and \( x'(t) \neq 0 \).
GROUP WORK
Find parametrizations for the given curves.

1. 

2. 

3. 

4. 

5. 

6. 

Segment of parabola
Velocity and Acceleration of Particles

**DE{\text{F}}\text{INITION:** velocity vector and acceleration vector

Given a position vector $\vec{x}$ as a function of a parameter $t$ for a particle, the velocity vector $\vec{v}$ is given by $\frac{d\vec{x}}{dt}$ with the speed being $||\vec{v}||$ and the acceleration vector $\vec{a}$ is given by $\frac{d\vec{v}}{dt}$.

In other words, if $\vec{x}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}$, then

$$\vec{v} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}$$

and

$$\vec{a} = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j} + \frac{d^2z}{dt^2} \hat{k}$$

Sometimes $\vec{v}$ and $\vec{a}$ are written $\dot{x}$ and $\ddot{x}$, respectively.

**Distance Travelled**

The length of a curve given by a position vector $\vec{x}(t)$ defined for an interval $a \leq t \leq b$ if the derivative $\frac{d\vec{x}}{dt} \neq 0$ in the interval $a < t < b$ is given by

$$\int_a^b ||\frac{d\vec{x}}{dt}|| \ dt \quad \text{or} \quad \int_a^b ||\vec{v}|| \ dt$$

**Circular Motion**

Given that $\vec{x}(t) = R \cos(\omega t) \hat{i} + R \cos(\omega t) \hat{j}$, this represents motion in a circle of radius $R$ with period $\frac{2\pi}{|\omega|}$ where $\vec{v}$ is tangent to the circle and speed is constant $|\omega|R$ and acceleration is $\omega^2 R$ pointed towards the center of the circle.

**EXAMPLE**

Let’s show the features of circular motion given above are true.