Worksheet 11

TITLE Second Order Partial Derivatives
CURRENT READING McCallum, Section 14.7
HW #6 (DUE WEDNESDAY 10/08/14)
McCallum, Section 14.6: 4, 11, 12, 26, 34, 35, 47*.
McCallum, Section 14.7: 6, 7, 8, 12, 19, 24, 30, 31, 41*.
McCallum, Section 14.8: 3, 12, 19*.
McCallum, Chapter 14: 2, 14, 35, 45, 64*

SUMMARY
This worksheet discusses higher order partial derivatives of multivariable functions and introduces the concept of the mixed partial derivative.

RECALL: second derivative
Given an infinitely differentiable function $y = f(x)$ its derivative $f'(x)$ represents the slope of the graph of the function at any point and $f''(x)$ represents the concavity of the graph. Also, $f'(x)$ represents the instantaneous rate of change of $f(x)$ at a point while $f''(x)$ represents the instantaneous rate of change of $f'(x)$.

The Second-Order Partial Derivatives of $f(x, y)$

DEFINITION: $f_{xx}$, $f_{xy}$, $f_{yy}$ and $f_{yx}$
Given a function $z = f(x, y)$ with continuous partial derivatives we can not only find the rate of change with $f$ with respect to $x$ and the rate of change of $f$ with respect to $y$ but the rate of change of those functions with respect to $x$ and $y$ also!

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx} \quad \text{and} \quad \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = (f_y)_y = f_{yy} \quad \text{and} \quad \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = (f_y)_x = f_{yx}$$

These expressions above are referred to as the second order partial derivatives of $f(x, y)$

EXAMPLE
McCallum, page 812, Exercise 4.
Compute the four second-order partial derivatives of $f(x, y) = e^{2xy}$

QUESTION: Do you notice a relationship between $f_{xy}$ and $f_{yx}$?
When Mixed Partial Derivatives Are Equal

**Theorem**
(Clairault’s Theorem) If \( f_{yx} \) and \( f_{xy} \) are continuous at some point \((a, b)\) found in a disc \((x - a)^2 + (y - b)^2 \leq D\) for some \(D > 0\) on which \(f(x, y)\) is defined, then \(f_{xy}(a, b) = f_{yx}(a, b)\).

Applications of the Second-Order Partial Derivatives
Recall (from Worksheet #8) that the local linearization of a function \(f(x, y)\) near the point \((a, b)\) is given by the tangent plane

\[
f(x, y) \approx P(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]  

(1)

**Taylor Polynomial Approximations**

Note that the expression on the right hand side of (1) can be thought of as Taylor Polynomial of Degree 1 approximating \(f(x, y)\) near \((a, b)\) for a function that has continuous first-order partial derivatives.

We can expand this idea from (1) to improve our approximation of this function. If \(f(x, y)\) has continuous second-order partial derivatives we can produce a Taylor Polynomial of Degree 2 approximating \(f(x, y)\) near \((a, b)\):

\[
f(x, y) \approx Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{f_{xx}(a, b)}{2}(x - a)^2 \\
+ f_{xy}(a, b)(x - a)(y - b) + \frac{f_{yy}(a, b)}{2}(y - b)^2
\]

**Exercise**

McCallum, page 811, Example 5.

Find the Taylor Polynomial of degree 2 at the point \((1,2)\) for the function \(f(x, y) = \frac{1}{xy}\).
You are told that there is a function $f$ whose partial derivative $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Do you believe this? PROVE YOUR ANSWER!

The kinetic energy of a body with mass $m$ and velocity $v$ is $K = \frac{1}{2}mv^2$. Show that $\frac{\partial K}{\partial m} \frac{\partial^2 K}{\partial v^2} = K$.

The gas law for fixed mass $m$ of an ideal gas at the absolute temperature $T$, pressure $P$ and volume $V$ is $PV = mRT$ where $R$ is the gas constant. Show that

$$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$$