TITLE The Directional Derivative and the Gradient Vector

CURRENT READING McCallum, Section 14.4 and 14.5.

HW #4 (DUE WED 09/24/14)
McCallum, Section 12.6: 24, 28, 35, 40, 52.
Section 14.1: 10, 11, 12, 13, 17, 18, 25, 26, 36, 37, 41, 48, 49.
Section 14.2: 8, 9, 14, 24, 25, 30, 34, 36, 39, 51, 52, 65*.

SUMMARY
This worksheet discusses the concepts of the directional derivative, i.e. the rate of change of a multivariable function in the direction of a given unit vector, as well as the gradient vector $\nabla f$.

The Directional Derivative

WARM-UP
Suppose we have a point $P$ at $(a,b)$ and we move a distance $h$ in the direction $\hat{u} = u_1 \hat{i} + u_2 \hat{j}$ to reach point $Q$. What are the coordinates of the point $Q$? DRAW A PICTURE!

QUESTION Suppose $z = f(x,y)$ was a function with $P$ and $Q$ in its domain how could we estimate the rate of change of $f$ from $P$ to $Q$?

ANSWER

DEFINITION: directional derivative
The rate of change of the function $f(x,y)$ in the direction of the unit vector $\vec{u} = u_1 \hat{i} + u_2 \hat{j}$ at the point $(a,b)$ is given by

$$f_{\vec{u}}(a,b) = \lim_{h \to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a, b)}{h}$$

(1)

The directional derivative given in Equation (1) is sometimes denoted $D_{\vec{u}}(f)$.

NOTE:
- If $\vec{u} = \hat{i}$ (i.e. $u_1 = 1$ and $u_2 = 0$) then the equation in (1) would just represent the partial derivative $f_x(a,b)$, i.e. the rate of change in the positive $x$ direction.
- If $\vec{u} = \hat{j}$ (i.e. $u_1 = 0$ and $u_2 = 1$) then the equation in (1) would just represent the partial derivative $f_y(a,b)$, i.e. the rate of change in the positive $y$ direction.
- The directional derivative is a SCALAR quantity!
Understanding the Directional Derivative Graphically

**GROUP WORK**

Consider the following contour diagrams for the functions $f(x, y)$, $g(x, y)$ and $h(x, y)$. Determine whether the directional derivative at the indicated point is positive, negative or zero in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$ and in the direction of $\vec{w} = 2\vec{i} + \vec{j}$.

![Contour Diagrams](image)

**Computing Directional Derivatives**

We already know how to compute $f_x$ and $f_y$ which are directional derivatives in the $\vec{i}$ direction and $\vec{j}$ direction, respectively.

Since every vector $\vec{u}$ in $\mathbb{R}^2$ can be written as a linear combination of a vector in the $\vec{i}$ direction and a vector in the $\vec{j}$ direction it should not be surprising that every directional derivative can be written as a linear combination of the partial derivatives of $f(x, y)$.

In other words, the directional derivative of $f(x, y)$ at $(a, b)$ in the direction $\vec{u} = u_1\vec{i} + u_2\vec{j}$ can be written

$$f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = (f_x(a,b)\vec{i} + f_y(a,b)\vec{j}) \cdot (u_1\vec{i} + u_2\vec{j})$$

(2)

**PROOF**

Let’s prove the result in (2), i.e. every directional derivative $f_{\vec{u}}(a,b)$ is the dot product of the vector of partial derivatives $f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$ with a direction vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$.

**Exercise**

McCallum, Page 781, Example 3. Calculate the directional derivative of $f(x, y) = x^2 + y^2$ at (1,0) in the direction $\vec{i} + \vec{j}$. 
The Gradient Vector

The result in (2) we proved is VERY useful because the vector composed of the partial derivatives of \( f(x, y) \), i.e. \( f_x(a, b)\vec{i} + f_y(a, b)\vec{j} \) is known as the **gradient vector** of \( f(x, y) \) at the point \((a, b)\). The gradient vector is sometimes written as \( \nabla f(a, b) \) or \( \text{grad} f(a, b) \).

This means that we can re-write (2) to produce the formula we will use to compute directional derivatives from now on.

\[
f_{\vec{u}}(a, b) = \nabla f(a, b) \cdot \vec{u} \quad \text{or} \quad f_{\vec{u}}(a, b) = \text{grad} f(a, b) \cdot \vec{u}
\]

(3)

**Interpreting The Meaning Of The Gradient Vector**

The directional derivative \( f_{\vec{u}}(a, b) \) equals the magnitude of the gradient vector \( ||\nabla f(a, b)|| \) times cosine of the angle \( \theta \) between \( \vec{u} \) and the direction the gradient vector points in (by definition of the dot product)!

\[
f_{\vec{u}}(a, b) = \nabla f(a, b) \cdot \vec{u} = ||\nabla f(a, b)|| ||\vec{u}|| \cos \theta = \frac{||\nabla f(a, b)||}{||\vec{u}||} \cos \theta
\]

(4)

Thus we can see from Equation (4) and the figure on this page that the rate of change of \( f \) in the direction \( \vec{u} \) will be the largest when the angle between \( \nabla f(a, b) \) and \( \vec{u} \) is 

**The Gradient Vector Points In The Direction Of Maximum Increase Of \( f(x, y) \)**

This means that the gradient vector \( \text{grad} f(a, b) \) always points in the **direction in which the functions changing the most**, also known as “the direction of maximum rate of increase of \( f \) at the point \((a, b)\),”

**The Gradient Vector Is (Always) Perpendicular To The Contours Of \( f(x, y) \)**

The direction in which the function \( f \) will have the smallest rate of change in magnitude (i.e no change or the function will be constant) will occur when the angle between \( \vec{u} \) and the gradient vector is 

By definition, contours or level sets are curves along which the function \( f \) is constant, i.e. does not change, so since the gradient vector points in the direction of maximum change, it is always orthogonal to the direction of zero change. Thus the gradient vector always points in a direction perpendicular to the contours of \( f \)!
Adapted from McCallum, page 786, Problems 56-60. Use the figure above to calculate estimates of

(a) \( f_\vec{u}(4, 1) \) where \( \vec{u} = \vec{i} \)

(b) \( f_\vec{u}(4, 1) \) where \( \vec{u} = \vec{j} \)

(c) \( f_\vec{u}(4, 1) \) where \( \vec{u} = (\vec{i} - \vec{j})/\sqrt{2} \)

(d) \( f_\vec{u}(4, 1) \) where \( \vec{u} = (-\vec{i} + \vec{j})/\sqrt{2} \)

(e) \( f_\vec{u}(4, 1) \) where \( \vec{u} = (-2\vec{i} + \vec{j})/\sqrt{5} \)