Class 3: Wednesday September 3

TITLE The Cross Product and Vector Projection

CURRENT READING McCallum, Section 13.3 and 13.4

HW #1 (DUE WED 09/03/14)
McCallum, Section 13.1: 12, 23, 26, 31, 37; Section 13.2: 8, 26, 33, 43-48.

SUMMARY
In today’s class we will be introduced to two more operations on vectors: the projection of one vector onto the direction of another, and the cross product of two vectors.

Vector Projection
For any vectors \( \vec{u} \) and \( \vec{v} \) where \( \vec{u} \neq \vec{0} \) then the projection of \( \vec{v} \) onto \( \vec{u} \) is the vector \( \text{proj}_{\vec{u}}(\vec{v}) \) defined by:

\[
\text{proj}_{\vec{u}}(\vec{v}) = \left( \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u}
\]  

Exercise
Draw a picture of the projection of \( \vec{v} \) onto \( \vec{u} \) in the space below:

One can think of a vector \( \vec{v} \) being split into two vectors, one vector called \( \vec{v}_{\text{perp}} \) that is perpendicular to the vector \( \vec{u} \) and another \( \vec{v}_{\text{parallel}} \) that is parallel to the vector \( \vec{u} \).

\[
\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\text{perp}} = \text{proj}_{\vec{u}}(\vec{v}) + \text{perp}_{\vec{u}}(\vec{v})
\]  

From Equation (2) it is clear that if one knows \( \vec{v}_{\text{parallel}} \) or \( \text{proj}_{\vec{u}}(\vec{v}) \) then one can compute \( \vec{v}_{\text{perp}} \) or \( \text{perp}_{\vec{u}}(\vec{v}) \) simply by subtraction.

\[
\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}} \\
\text{perp}_{\vec{u}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{u}}(\vec{v})
\]
EXAMPLE
Let’s look at the derivation of the projection formula in Equation (1) given that we know that the angle between \( \vec{v} \) and \( \vec{u} \) is \( \theta_{uv} \).

\[
\vec{p} = ||\vec{v}|| \cos \theta_{uv} \hat{u} \\
= ||\vec{v}|| \left( \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||} \right) \left( \frac{\vec{u}}{||\vec{u}||} \right) \\
= \left( \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||^2} \right) \vec{u} \\
= \left( \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u} \\
\vec{p} = \text{proj}_{\vec{u}}(\vec{v})
\]

NOTE
- The McCallum textbook uses the notation \( \vec{v} \parallel \) instead of \( \text{proj}_{\vec{u}}(\vec{v}) \)
- The projection of a vector onto another is a vector!

QUESTION What is the difference between \( \text{proj}_{\vec{u}}(\vec{v}) \) and \( \text{proj}_{\vec{v}}(\vec{u}) \)?

GROUP WORK
McCallum, page 742, Exercise #42. Given \( \vec{v} = 3\vec{i} + 4\vec{j} \) and force vector \( \vec{F} = 4\vec{i} + \vec{j} \), find:
(a) The component of \( \vec{F} \parallel \vec{v} \).
(b) The component of \( \vec{F} \perp \vec{v} \).
(c) The work \( W \) done by \( \vec{F} \) through displacement \( \vec{v} \).
The Vector Cross Product
In addition to the dot product between two given vectors, there is another important binary operation one can perform on two vectors called the **vector cross product**.

**DEFINITION**
The **cross product** of two vectors \( \vec{u} = (u_1, u_2, u_3) \) and \( \vec{v} = (v_1, v_2, v_3) \) in \( \mathbb{R}^3 \) is defined to be the vector

\[
\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)
\]

(3)

Luckily, there’s an easy way to remember the cross product calculation (3) as the determinant of a 3x3 matrix

\[
\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}
\]

**EXAMPLE**
Find the cross product \( \vec{u} \times \vec{v} \) of \( \vec{u} = (1, -3, 2) \) and \( \vec{v} = (2, 4, -5) \).

**Exercise**
Take the dot product of our answer with both \( \vec{u} \) and \( \vec{v} \). Do you notice anything remarkable?

**Properties of The Vector Cross Product**

\[
\vec{u} \cdot (\vec{u} \times \vec{v}) = 0 \\
\vec{v} \cdot (\vec{u} \times \vec{v}) = 0 \\
\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \\
\hat{i} \times \hat{j} = \hat{k}; \hat{j} \times \hat{k} = \hat{i}; \hat{k} \times \hat{i} = \hat{j}
\]

**Additivity:** \( \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \) and \( (\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w} \)

**Associativity:** \( r(\vec{u} \times \vec{v}) = (r\vec{u}) \times \vec{v} = \vec{u} \times (r\vec{v}) \)
Geometric Applications of The Vector Cross Product

The vector cross product can be used to solve a number of interesting geometric problems.

Finding The Equation Of A Plane Through Three Points

Given three points that lie on a plane in $\mathbb{R}^3$, i.e. $\vec{P} = (p_1, p_2, p_3)$, $\vec{Q} = (q_1, q_2, q_3)$ and $\vec{R} = (r_1, r_2, r_3)$ we can find the normal vector for the plane that these point lie in by taking the cross product between the displacement vectors between any two pairs of points:

$\vec{n} = \vec{PQ} \times \vec{PR}$ or $\vec{n} = \vec{QR} \times \vec{QP}$ or $\vec{n} = \vec{RP} \times \vec{RQ}$

Once you know the normal vector you can simply write down the general equation of the plane using the formula $\vec{n} \cdot \vec{p} = \vec{n} \cdot \vec{p}_0$ where $\vec{p}$ is a known point on the plane.

**EXAMPLE**

Show that the equation of the plane through the points (1,3,0), (3,4,-3) and (3,6,2) is equal to $11x - 10y + 4z = -19$.

Area of a Parallelogram

The vector crossproduct can be used to find the area of the parallelogram that has $\vec{a}$ and $\vec{b}$ as its sides.

$$\text{Area of Parallelogram bounded by } \vec{a} \text{ and } \vec{b} = ||\vec{a} \times \vec{b}|| = ||\vec{a}|| \cdot ||\vec{b}|| \sin \theta_{ab}$$

Volume of a Parallelepiped

A parallelepiped is a 3-dimensional object formed by having a parallelogram on each of its six sides. One can form a parallelepiped from three non-parallel vectors in $\mathbb{R}^3$. (If the three vectors are parallel and of the same magnitude, the object formed would be a cube).

$$\text{Volume of Parallelepiped bounded by } \vec{a}, \vec{b} \text{ and } \vec{c} = |\vec{c} \cdot (\vec{a} \times \vec{b})|$$