#1 Chain Rule, Partial Derivatives. (a) In this problem you have \( x = s + t \) and \( y = s - t \) so that \( f(x, y) \) is really \( f(x(s, t), y(s, t)) \). Everyone was able to draw the correct relationship between the variables. (b) Then use the diagram to write down the correct chain rule expression for \( f_s = f_x x_s + f_y y_s \) and \( f_t = f_x x_t + f_y y_t \). (c) Use the information from above to show that \( x_s = 1 \), \( x_t = 1 \), \( y_s = 1 \) and \( y_t = -1 \). Multiple your expressions for \( f_s \) and \( f_t \) together and you will see it becomes \( f_x^2 - f_y^2 \).

#2 Unconstrained Multivariable Optimization, Extreme Value Theorem, Repeated Partial Differentiation. (a) The function is \( f(x, y) = x^3 - xy - y^2 + y \) so that \( f_x = 3x^2 - y \) and \( f_y = -x - 2y + 1 \). To find critical points one has to find the points \( (x, y) \) so that both \( f_x = 0 \) and \( f_y = 0 \) are satisfied simultaneously! This is NOT the same things as saying simply that \( f_x = f_y \). The critical points end up being \((-\frac{1}{3}, \frac{2}{3})\) and \((\frac{1}{3}, \frac{4}{3})\). By checking the expression \( D = f_{xx}f_{yy} - f_{xy}^2 \) at these points one can see that \( D(-\frac{1}{3}, \frac{2}{3}) = 5 > 0 \) which indicates a local max (since \( f_{xx} < 0 \) at this point). \( D(\frac{1}{3}, \frac{4}{3}) = 5 < 0 \) which indicates a saddle. (b) A saddle obviously can not be a global extrema. So the only candidate is the local max but since it is clear that when \( x \to \infty \) and \( y = 0 \), \( f(x, y) \to -\infty \) the local max is not the global max. The extreme value theorem tells you that if the domain is closed and bounded THEN you must have a global max and global min. It doesn’t tell you anything if the domain is NOT closed or NOT bounded.

#3 Constrained Multivariable Optimization, Lagrange Multipliers. The key problem here is to figure out that the constraint is the function \( g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) while the objective function is the area of the rectangle trapped within the ellipse, i.e. \( f(x, y) = 4xy \). First thing to do is to compute \( f_x \), \( f_y \), \( g_x \) and \( g_y \). This get you the Lagrange Multiplier equations \( f_x = 2y = \lambda g_x = \lambda \frac{2x}{a^2} \), \( f_y = 2x = \lambda g_y = \lambda \frac{2y}{b^2} \), \( g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) These first two can be manipulated to obtain the expression \( x^2b^2 = y^2a^2 \) This means that \( \frac{y^2}{b^2} = \frac{x^2}{a^2} \) so combining with the constrain equation gives you \( \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \) so \( y^2 = b^2/2 \) and \( x^2 = a^2/2 \) which means that \( x^2y^2 = a^2b^2/4 \) so \( xy = ab/2 \) and the maximum area \( 4xy = 2ab \).

#4 Polar Coordinates, Iterated Integration, Multiple Integration. (a) The integral \( \mathcal{I} = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y \, dx \, dy \) which means that \( 0 \leq y \leq 1 \) and \(-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \). This means that \( x^2 + y^2 = 1 \). So the region being integrated is the top half of the unit disk centered at the origin. (b) and (c) You need to evaluate two of the following three integrals.

\[
\mathcal{I} = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y \, dx \, dy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} y \, dx \, dy = \int_0^\pi \int_0^1 (r \sin \theta) \, r \, dr \, d\theta = \frac{2}{3}.
\]

BONUS Triple Integral. The volume of a cone is \( \frac{1}{3} \pi R^2 h \) (best choice is to use cylindrical coordinates). The volume of the triangular pyramid with base \( \frac{1}{2}ab \) is \( \frac{1}{6} \pi ab h \) (best choice is to use double integral of \( z = h(1 - x/a - y/b) \)).