4. (20 points) Multiple Integration.

a. (10 points) Evaluate $\int \int_R ye^x \, dA$ where $R$ is the first quadrant of the circle of radius 4 centered at the origin. (Sketch the region $R$).

b. (10 points) Consider $\int_0^1 \int_0^1 \int_{x+y}^{1-y} dz \, dy \, dx = \frac{1}{12}$. Re-compute this integral using a different triple integral which represents the same volume.
3. (20 points) Iterated Integration.

a. (10 points) Evaluate \( \int_{-3}^{0} \int_{0}^{2} \int_{-1}^{1} \cos(x + y + z) - xyz \, dx \, dz \, dy \)

b. (10 points) Evaluate \( \int_{1}^{2} \int_{0}^{\ln z} \frac{1}{x} \, dy \, dx \)
5. (20 points) Constrained Multivariable Optimization, Lagrange Multipliers
The “geometric mean” of \( n \) numbers is defined as \( f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 x_3 \ldots x_n} \). Suppose that \( x_1, x_2, \ldots, x_n \) are positive numbers such that \( \sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n = c \), where \( c \) is a constant.

a. (10 points) Find the maximum value of the geometric mean of \( n \) positive numbers given the constraint that their sum must be equal to a constant. [HINT: Consider \( f^n \) instead of \( f \)].

b. (10 points) You can deduce from part (a) that the geometric mean of \( n \) numbers is always less than or equal to the arithmetic mean, that is:

\[
\sqrt[n]{x_1 x_2 x_3 \ldots x_n} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}
\]

Under what conditions will the geometric mean be exactly equal to the arithmetic mean of those same \( n \) numbers?
EXTRA CREDIT (10 points.) Unconstrained Multivariable Optimization

Consider $f(x, y) = x^4 + y^4 - 4xy + 1$.

a. (5 points) Find the three critical points of $f(x, y)$.

b. (5 points) Use the Second Derivative Test to classify each of the three critical points of $f(x, y)$.
(e) Using the picture alone, estimate the points at which the objective function \( f(x, y) \) achieves a global minimum on the constraint set \( g(x, y) = 0 \) and the values of \( f \) there. EXPLAIN YOUR ANSWER.
(f) Use the Method of Lagrange Multipliers to obtain the minimum value of
\[ f(x, y) = x^2 + xy + y^2 \]
on the constraint set \[ g(x, y) = x + y - 2 = 0. \]

(g) How would the maximum and minimum on the constraint set change if the constraint set \( g(x, y) \) were changed to \( h(x, y) = x^2 + y^2 - 4 \)? Find the extrema of \( f(x, y) = x^2 + xy + y^2 \) subject to the constraint \( h(x, y) = 0 \) and EXPLAIN YOUR ANSWER.
2. (20 points.) Multiple Integration.

The goal of this question is to evaluate \( \int_0^\infty e^{-x^2} \, dx = \lim_{a \to \infty} \int_0^a e^{-x^2} \, dx. \)

(a) (10 points.) Find \( I(R) = \iint_{D_R} e^{-(x^2+y^2)} \, dx \, dy \) when \( D_R \) is \( x^2 + y^2 \leq R^2 \) (the interior of the circle of radius \( R \) centered at the origin). **HINT:** pick a useful coordinate system!

(b) (5 points.) Take your answer \( I(R) \) to (b) and then let \( R \to \infty \). What is \( \lim_{R \to \infty} \iint_{D_R} e^{-(x^2+y^2)} \, dx \, dy \)?

(c) (5 points.) Given that \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 \) then what is the value of \( \int_0^\infty e^{-x^2} \, dx \)?
4. (20 points.) Iterated Integration.

Consider the iterated integral for $V = \int_{-1}^{1} \int_{0}^{x^2} \int_{-1}^{1} dz \, dy \, dx = \frac{4}{3}$

(a) (12 points.) Write down 3 (three) of the 5 (five) other possible triple iterated integrals which represent the exact same value $V$. HINT: There is no dependence of $z$ upon $y$) DO NOT EVALUATE THESE INTEGRALS.

(b) (8 points.) Use any one of the iterated integrals you wrote down in part (a) to confirm the value of $V$. 
2. Multivariable Chain Rule. 25 points.

Consider the functions \( u(x, y, z) = f(x - y, y - z, z - x) \). Our goal is to show that a function \( u \) with this form satisfies the following famous partial differential equation

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.
\]

(a) (10 points.) Consider a function \( u = f(r, s, t) \) where \( r = r(x, y, z) \), \( s = s(x, y, z) \) and \( t = t(x, y, z) \) are given. In other words, although \( u \) is a function of \( r, s \) and \( t \), since each of these functions is a function of \( x, y \) and \( z \) one can consider \( u \) as a function of \( x, y \) and \( z \).

Use the Chain Rule to write down expressions for \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial y} \), and \( \frac{\partial u}{\partial z} \). [HINT: draw a “tree diagram” reflecting the relationships between the variables to assist you.]

(b) (15 points.) Let \( r = x - y, s = y - z \) and \( t = z - x \). Use this information and your answer to (a) to show that \( u(x, y, z) = f(x - y, y - z, z - x) \) satisfies the equation \( u_x + u_y + u_z = 0 \).
5. (20 points.) Constrained Multivariable Optimization, Lagrange Multipliers
Recall the Cobb-Douglas function \( P(L,K) = bL^\alpha K^{1-\alpha} \) where the total production \( P \) of a certain product depends on the amount of labor \( L \) used and the amount \( K \) of capital investment (\( 0 < \alpha < 1 \) and \( b > 0 \)).
If the cost of a unit of labor is \( m \) and the cost of unit of capital is \( n \), given that the production of the company is fixed at a level \( Q \), what values of \( L \) and \( K \) will minimize the cost function \( C(L,K) = mL + nK \)?

a. (10 points) Write down the equations you need to solve simultaneously to find the answer to the question.

b. (10 points) Solve the equations to find the values of \( L \) and \( K \) which minimize the cost function \( C(L,K) \). (HINT: Eliminate the Lagrange Multiplier first).