Multivariable Calculus

Class 29: **Monday April 24**

SUMMARY Green's Theorem
CURRENT READING Williamson & Trotter, §9.1
HOMEWORK page 408: **3, 4, 6, 7, 10 Extra Credit page 409: 15, 18, 20**

THEOREM: Green's Theorem

Given a planar region D whose boundary is a single closed curve γ parametrized by a function $\vec{g}(t)$ so that as t increases from a to b, $\vec{g}(t)$ traces out γ once in the *counter-clockwise* direction, then if $F(x, y)$ and $G(x, y)$ are real-valued functions defined on D including its boundary, then the formula for Green' Theorem is:

$$\int\int_D \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx\ dy = \oint_\gamma F\ dx + G\ dy \qquad \textbf{(Green's Theorem)}$$

EXAMPLE 1

Williamson & Trotter, page 408, #1. Use Green's Theorem to compute the value of the line integral $\oint_\gamma y\ dx + x^2\ dy$ where γ is the indicated path.
γ is the circle given by $g(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Exercise 1

Williamson & Trotter, page 408, #2. Repeat problem #1, this time with γ being the square with corners at $(\pm 1, \pm 1)$, traced counter-clockwise.

EXAMPLE 2

Williamson & Trotter, page 408, #11. Show that if D is a simple region bounded by a piecewise smooth curve γ, traced counter clockwise, then the area of the interior of γ (i.e. the area of D) is given by

$$A(D) = \frac{1}{2}\oint_\gamma -y\ dx + x\ dy$$

Now that we know about **curl** and **div** we can use them to re-write the Green's Theorem result. Note that the integrand in the area integral in Green's Theorem is the non-zero component in $\vec{\nabla} \times \vec{F}$ where $\vec{F}(x,y) = (F(x,y), G(x,y), 0)$

THEOREM: Stokes' Theorem in the Plane

So, in vector format, we can write Green's Theorem as

$$\int\int_D (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\ dA = \oint_{\partial D} \vec{F} \cdot d\vec{x}$$

This result is known as **Stokes' Theorem in the Plane**. Note that $d\vec{x} = \hat{t}ds$ where s is a parameter representing arc length along the curve γ (not important) in a direction $\hat{t}$ always tangent to the curve.

In Stokes' Theorem we are dealing with a vector field in $\mathbb{R}^2$ which has the form $\vec{F} = (F(x,y), G(x,y))$. In Gauss' Theorem we are dealing with a vector field which has the form $\vec{H} = (-G(x,y), F(x,y))$

Suppose we had a unit vector $\hat{n}$ which is orthogonal in the plane (i.e. normal) to the curve γ at every point. This means that $\hat{n}$ is at 90 degrees (orthogonal) to the tangent vector $\hat{t}$ and points **away** from the interior of a closed region D where γ is a path which makes up the boundary of D. Mathematically, then $\hat{n} \cdot \hat{t} = 0$.

We can show that $\vec{H} \cdot d\vec{x} = \vec{F} \cdot \hat{n}\ ds$. (Please note that $d\vec{x} = (dx, dy)$ and $ds\ \hat{n} = (-dy, dx)$ so that $(ds)^2 = (dx)^2 + (dy)^2 = |d\vec{x}|^2$. Can you visualize this?)

Since $\vec{H} \cdot d\vec{x} = (-G, F) \cdot (dx, dy) = -G\ dx + F\ dy = (F, G) \cdot (dy, -dx) = \vec{F} \cdot \hat{n}\ ds$

But using Green's Theorem on $\vec{H} = (-G(x,y), F(x,y))$ and recalling $\vec{F} = (F(x,y), G(x,y))$ produces

$$\begin{aligned}
\int_\gamma \vec{H} \cdot d\vec{x} = \int_\gamma -G(x,y)\ dx + F(x,y)\ dy &= \int\int_D \left(\frac{\partial}{\partial x}[F(x,y)] - \frac{\partial}{\partial y}[-G(x,y)]\right) dA \\
\int_\gamma \vec{F} \cdot \hat{n}\ ds &= \int\int_D \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dA
\end{aligned}$$

But the right hand side of this expression should remind you of the divergence operator. So we can also re-write Gauss' theorem using another differential vector operator, this time **div** $\vec{F}$

THEOREM: Gauss' Theorem in the Plane

$$\int\int_D \vec{\nabla} \cdot \vec{F}\ dA = \int_{\partial D} \vec{F} \cdot \hat{n}\ ds$$

THEOREM: Divergence Theorem

$$\int\int_{\partial D} \vec{F} \cdot d\vec{A} = \int\int\int_{D} \vec{\nabla} \cdot \vec{F} \; dV$$

The divergence theorem is the most general form or the Gauss' Theorem, equating the integral of the divergence of a vector field $\vec{F}(\vec{x})$ through a volume of space D to the surface area integral over the boundary of the region, called ∂D.

EXAMPLE 1

Willliamson & Trotter, Page 408, # 9. Evaluate $\int_C (x^2 - y^2) \; dx + (x^2 + y^2) \; dy$ where C is the circle of radius 1 centered at the origin and traced clockwise.

Exercise 1

Willliamson & Trotter, Page 408, # 12. Let f be a real valued function with continuous second order derivatives in an open set D in $\mathbb{R}^2$. Let $\vec{F}$ a vector field defined in D by $\vec{F} = \vec{\nabla} f(\vec{x})$. Show that if $\vec{F} = (F(\vec{x}), G(\vec{x}))$, then the equation $\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = 0$ at all points in region D.