Multivariable Calculus

Math 212 Spring 2006 — Fowler 112 MWF 8:30pm - 9:25am

 — http://faculty.oxy.edu/ron/math/212/06/

Class 27: **Monday April 17**

SUMMARY Introduction to Path Integrals
CURRENT READING Williamson & Trotter, §8.1
HOMEWORK Williamson & Trotter, page 376: 1, **2,8,9,14 Extra Credit page 376: 19** AND page 376-377: 22,25,**28,29**,30 **Extra Credit page 376: 33**

DEFINITION: path integral

Given a vector function $\vec{f}:\mathbb{R}^n \to \mathbb{R}^n$ and a path or curve γ in space given by $\vec{g}(t):\mathbb{R}\to\mathbb{R}^n$ the path integral of $\vec{f}$ over $\gamma$is given by $\displaystyle\int_a^b \vec{f}(\vec{g}(t))\cdot\frac{d\vec{g}}{dt}\,dt = \int_\gamma \vec{f}\cdot d\vec{x}$

EXAMPLE 1

Williamson & Trotter, page 371, example 4 Given the vector field $\vec{F}(\vec{x}) = (x-y, y-z, z-x)$ and the curve γ given $\vec{g}(t) = (t, -t, t^2)$ for $0 \le t \le 1$. Compute the line integral of $\vec{F}$ over γ.

Exercise 1

Williamson & Trotter, page 376, #3. Compute $\displaystyle\int_{\gamma_1} x\ dy$ and compute $\displaystyle\int_{\gamma_2} x\ dy$ where γ_1 is given by $\vec{g}(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$ and γ_2 is given by $\vec{h}(t) = (\cos t, \sin t)$ for $0 \le t \le 4\pi$.

GROUPWORK

Williamson & Trotter, page 376, #7. Compute $\int_\gamma \vec{F} \cdot d\vec{x}$ where $\vec{F}(\vec{x}) = (z, x, y)$ and γ is given parametrically by $(x, y, z) = (\cos t, \sin t, t)$

THEOREM: Fundamental Theorem of Calculus

Given that f is a continuously differentiable real-valued function defined in an open subset D of $\mathbb{R}^n$. (Then we know $\vec{\nabla} f$ is a continuous vector field in D called a **gradient field**.) IF γ is a smooth curve in D with initial point $\vec{a}$ and terminal point $\vec{b}$ THEN $\displaystyle\int_\gamma \vec{\nabla} f \cdot d\vec{x} = f(\vec{b}) - \vec{f}(\vec{a})$.

In other words, the line integral of a gradient field over a curve only depends on the value of the function at the endpoints of the curve. This also implies that in a gradient field if γ is any closed curve or loop (i.e. $\vec{a} = \vec{b}$), then the line integral over γ will be zero!

THEOREM: Fundamental Theorem of Calculus, part 2

Given a gradient field $\vec{F}$, the solution to the vector equation $\vec{\nabla} f = \vec{F}(\vec{x})$ is $f(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{t}) \cdot d\vec{t}$ with $f(\vec{x}_0) = 0$.

This is an important idea in physics in that we are often looking for what is called a **potential function** $\phi(\vec{x})$ which describes the behavior of a known vector field $\vec{F}$ such that $\vec{\nabla}\phi = \vec{F}$.

EXAMPLE 2

Williamson & Trotter, page 395, #14. Define a vector field as $\vec{F}(\vec{x}) = \vec{x}$ and parametrize the line segment γ joining $\vec{a}$ and $\vec{b}$ by $\vec{x}(t) = t\vec{b} + (1 - t)\vec{a}$ with $0 \leq t \leq 1$.

(a) Show by direct computation that $\displaystyle\int_\gamma \vec{F} \cdot d\vec{x} = \frac{1}{2}(|\vec{b}|^2 - |\vec{a}|^2)$

(b) Repeat the computation by finding a real-valued function f such that $\vec{\nabla} f = \vec{F}$ and applying the Fundamental Theorem of Calculus.