Class 24: **Monday April 10**

SUMMARY Multiple Integration
CURRENT READING Williamson & Trotter, Section 7.2
HOMEWORK #23 Williamson & Trotter, page 332: **1,2,9,11,13**, **Extra Credit page 333: 17**

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **bounded on a set** B if there exists a real number K such that $|f(\vec{x})| \leq K$ for all $\vec{x}$ in B.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined and bounded on a bounded subset B of the domain of f. Then we shall define $f_B(\vec{x}) = \begin{cases} f(\vec{x}), & \text{if } \vec{x} \text{ is in } B \\ 0, & \text{if } \vec{x} \text{ is NOT in } B \end{cases}$

Definition

The **content** V of a **coordinate rectangle** R is defined as the product $V(R) = \Pi_{i=1}^{n}(b_i - a_i)$ where a coordinate rectangle is a subset of $\mathbb{R}^n$ such that $a_i \leq x_i \leq b_i$ for $i = 1, 2, \ldots, n$.

Definition

The Riemann integral of $f : \mathbb{R}^n \to \mathbb{R}$ over B is denoted as $\int_B f dV$ and is defined by

$$\lim_{m(G)\to 0} \sum_{i=1}^{r} f_B(\vec{x}_i)V(R_i) = \int_B f dV$$

In the above definition $m(G)$ is a mesh with grids G covering the set B and $\vec{x}_i$ is a random point on one of r coordinate rectangles R_i with content $V(R_i)$ on the grids G. The point is that as the grids are defined so that the mesh becomes finer and finer (i.e. one approximates the set B with rectangles with smaller and smaller content $V(R_i)$) then in the limit this sum reaches a number, this number is the Riemann integral of f over B.

THEOREM

Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined and bounded on a bounded set B such that **(i)** the boundary of B has zero content and **(ii)** f is continuous except possibly on a set of zero content. THEN f is Riemann integrable over B.

Notation

Depending on whether $\vec{x}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$ the integral of $f(\vec{x})$ over $B \subset \mathbb{R}^n$ can be denoted $\displaystyle\int_B f\ dV$ or $\displaystyle\int_B f\ dx\ dy\ dz$ in $\mathbb{R}^3$ or $\displaystyle\int\int_B\int f\ dx\ dy\ dz$ in $\mathbb{R}^3$

$\displaystyle\int_B f dA$ or $\displaystyle\int_B f\ dx\ dy$ in $\mathbb{R}^2$ or $\displaystyle\int_B\int f\ dx\ dy\ dz$ in $\mathbb{R}^3$

EXAMPLE 1

Williamson & Trotter, page 332, # 8. Find the volume under the graph of f and above the set B where $f(x, y) = x + y + 2$ and B is the region bounded by the curves $y^2 = x$ and $x = 2$

Exercise 1

Williamson & Trotter, page 332, # 14. Write an expression for the volume of the ball $x^2 + y^2 + z^2 \leq a^2$ **(a)** as a triple integral and **(b)** as a double integral

THEOREMS

Linearity: $\int_B af(\vec{x}) + cg(\vec{x})\ dV = a\int_B f(\vec{x})\ dV + c\int_B g(\vec{x})\ dV$

Positivity: If $f \geq 0$ and integrable over B then $\int_B f\ dV \geq 0$

Union: $\int_{B_1 \cup B_2} f\ dV = \int_{B_1} f\ dV + \int_{B_2} f\ dV$

Leibniz Rule If $\partial g/\partial y$ is continuous for $a \leq x \leq b$ and $c \leq y \leq d$ then

$$\frac{d}{dy}\int_a^b g(x,y)\ dx = \int_a^b \frac{\partial g}{\partial y}(x,y)\ dx$$

Exercise 2

Williamson & Trotter, page 337, # 10. Find $g'(t)$ where $g(t) = \int_1^2 \frac{1}{x}e^{tx}\ dx$.

GroupWork

Williamson & Trotter, page 364, # 14. Evaluate $\int_S x^2y^2\ dx\ dy$ where S is the square $|x| + |y| \leq 1$.