Multivariable Calculus

Math 212 Spring 2006 — Fowler 112 MWF 8:30pm - 9:25am

 — http://faculty.oxy.edu/ron/math/212/06/

Class 23: **Friday April 7**

SUMMARY Introduction to Iterated Integration
CURRENT READING Williamson & Trotter, Section 7.1
HOMEWORK #22 Williamson & Trotter, page 321: 3,4,5,**6**,**7**,11,12,**16**,**17**,**21**
Extra Credit page 322: 24, 29

Recall that $\int_a^b f(x)dx$ is a CONSTANT. Let's consider a function $f(x, y)$ such that $f : \mathbb{R}^2 \to \mathbb{R}$ is defined on the rectangle $a \leq x \leq b \cap c \leq y \leq d$.
Consider $\int_a^b f(x, y)dx$ and $\int_c^d f(x, y)dy$.
Question: Are these objects constants?
Answer: No!
Thus we can evaluate iterated integral by integrating with respect to one variable and then the other, in sequence.

Fubini's Theorem

$$\begin{aligned} I = \int_a^b \int_c^d f(x, y)dydx = \int_a^b \left[\int_c^d f(x, y)dy \right] dx &= \int_a^b F(x)dx \\ = \int_c^d \int_a^b f(x, y)dxdy = \int_c^d \left[\int_a^b f(x, y)dx \right] dy &= \int_a^b G(y)dy = I \end{aligned}$$

EXAMPLE 1

Evaluate $\int_0^1 \int_1^2 x^2 + y \; dx \; dy$ two different ways to illustrate the result above.

What happens if the region of interest is non-rectangular?

Iterated Integration Over Non-Rectangular Regions

To integrate $f(x, y)$ over a "y-simple" region defined as $a \leq x \leq b \cap u(x) \leq y \leq v(x)$ use

$$\int_a^b \int_{u(x)}^{v(x)} f(x, y) dy dx$$

To integrate $f(x, y)$ over a "x-simple" region defined as $r(y) \leq x \leq s(y) \cap c \leq y \leq d$ use

$$\int_c^b \int_{r(y)}^{s(y)} f(x, y) dx dy$$

Exercise 1

Draw an example of a y-simple region and an x-simple region in the space below.

EXAMPLE 2

Integrate $f(x, y) = xy$ over the region bounded the vertical lines $x = -1$ and $x = 2$ and by the graphs $y = 1 + x^2$ and $y = -x^2$.

GroupWork

Williamson & Trotter, page 321, # 18. Sketch the region defined by $x \geq 0, x^2 + y^2 \leq 2$ and $x^2 + y^2 \geq 1$. Write down the integral over the region in each of the two possible orders of $f(x, y) = x^2$ and evaluate them.

EXAMPLE 3

Williamson & Trotter, page 321, # 15. Evaluate $\int_0^{\pi} \sin x \; dx \int_0^1 \; dy \int_0^2 (x + y + z) \; dz$