
Multivariable Calculus

Math 212 Spring 2006
©2006 Ron Buckmire

Fowler 112 MWF 8:30pm - 9:25am
<http://faculty.oxy.edu/ron/math/212/06/>

Class 22: Monday April 3

SUMMARY Curvilinear Coordinates

CURRENT READING Williamson & Trotter, Section 6.5

HOMEWORK Williamson & Trotter, page 308:3,4,6,7,9,10,12; page 310: 10,21,29,30,36,37

Extra Credit page 309: 15; page 310: 16, 33, 38

Polar Coordinates

Let $\vec{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a function which maps the 2-dimensional xy -plane into the 2-dimensional $r\theta$ -plane.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \vec{f}(r, \theta) = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix}, \text{ where } 0 < r < \infty, 0 \leq \theta < 2\pi$$

The image of \vec{f} is the entirety of \mathbb{R}^2 except the origin. The numbers r and θ are known as **polar coordinates**. The function \vec{f} also has an inverse function \vec{g} , which means we can map FROM polar coordinates back to cartesian xy -coordinates (but you have to be careful about inverting the trig functions)

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \vec{g}(x, y) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \arctan\left(\frac{y}{x}\right) + k\pi \end{bmatrix}, \text{ where } x \neq 0, k \in \mathbb{Z}$$

Spherical Coordinates

Let $\vec{s}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a function which maps the 3-dimensional xyz -plane into the 3-dimensional $r\phi\theta$ -plane.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{s}(r, \phi, \theta) = \begin{bmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{bmatrix}, \text{ where } 0 < r < \infty, 0 \leq \phi < \pi, 0 \leq \theta < \pi$$

Here the image of $\vec{s}(r, \phi, \theta)$ is all of \mathbb{R}^3 except the z -axis. The numbers r , ϕ and θ are known as **spherical coordinates**. Like with polar co-ordinates we can map FROM spherical coordinates TO cartesian coordinates by inverting \vec{s} , which we will call \vec{t} .

$$\begin{bmatrix} r \\ \phi \\ \theta \end{bmatrix} = \vec{t}(x, y, z) = \begin{bmatrix} \sqrt{x^2 + y^2 + z^2} \\ \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\ \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \end{bmatrix}, \text{ where } y \geq 0, x^2 + y^2 > 0$$

Notice that when $\phi = 0$ spherical coordinates are identical to polar coordinates.

Cylindrical Coordinates

Cylindrical coordinates are basically the three dimensional version of polar co-ordinates where a 3rd coordinate z is added.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \\ z \end{bmatrix}, \text{ where } 0 < r < \infty, -\pi < \theta \leq \pi, -\infty \leq z < \infty$$

EXAMPLE 1

Let's compute the Jacobian (derivative matrices) of the standard transformations to Cartesian coordinates from polar, spherical and cylindrical coordinates.

Exercise 1

Find the determinant of the Jacobian matrices of the standard polar, spherical and cylindrical coordinate transformations.

CHAPTER 6 REVIEW EXERCISES

Williamson & Trotter, page 309, #6. Find $\frac{\partial}{\partial x} f(K(x, y), K(x+y, x-y), x^2+y^2)$ where $f(x, y, z) = x^2 + y^2 - z^2$ and $K(x, y) = xy$.

Williamson & Trotter, page 311, #42. Find the points on $x^2 + 2xy + 3y^2 = 14$ which are closest to and furthest from the origin.