Multivariable Calculus

Math 212 Spring 2006 — Fowler 112 MWF 8:30pm - 9:25am

 — http://faculty.oxy.edu/ron/math/212/06/

Class 13: **Wednesday February 22**

SUMMARY Differentiability of a Vector Function of a Vector Variable: The Gradient
CURRENT READING Williamson & Trotter, Section (Section 5.2, 5.3)
HOMEWORK Williamson & Trotter, page 232: **6, 7**, 8, 9, 12, **16, 19, 20**;
Extra Credit page 232: # 21

Recall that the definition of a derivative
$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = A$ and $\lim_{x\to x_0} \frac{f(x)-f(x_0)-A(x-x_0)}{x-x_0} = 0$ where $f'(x_0) = A$.

DEFINITION: **differentiable**

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. f is **differentiable** at $\vec{x}_0$ if $\vec{x}_0$ is an interior point of the domain of f and there exists a vector $\vec{a}$ such that

$$\lim_{\vec{x}\to\vec{x}_0} \frac{f(\vec{x}) - f(\vec{x}_0) - \vec{a}\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|} = 0$$

The vector $\vec{a}$ is called the **gradient** of the differentiable function $\vec{f}(\vec{x})$ at $\vec{x}_0$ and is denoted $\vec{\nabla} f$ which is pronounced "del f" or "grad f."

DEFINITION:

If a function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}_0$ then the k^{th} coordinate of the gradient $\vec{\nabla} f(\vec{x}_0)$ is the k^{th} partial derivative of f at $\vec{x}_0$, for $k = 1, 2, \ldots, n$.

$$\vec{\nabla} f(\vec{x}_0) = (f_{x_1}, f_{x_2}, f_{x_3}, \ldots, f_{x_n})$$

Theorem

Let the domain of $f : \mathbb{R}^n \to \mathbb{R}$ be an open subset D of $\mathbb{R}^n$ for which all partial derivatives of $\frac{\partial f}{\partial x_i}$ are continuous. Then f is differentiable at every point of D.

EXAMPLE

Compute $\vec{\nabla} f$ for $f(x, y) = xy^2z^3$.

Theorem: Differentiability Implies Continuity

If a function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\vec{x}_0$ of its domain, then f is continuous at $\vec{x}_0$.

Tangent Approximation

For a function $f : \mathbb{R} \to \mathbb{R}$ we know that the tangent approximation is

$$T(x) = f(x_0) + f'(x_0)(x - x_0)$$

This is the first degree Taylor Polynomial approximation.
For a function $f : \mathbb{R}^n \to \mathbb{R}$ we can write the Tangent Approximation as

$$T(\vec{x}) = f(\vec{x}_0) + \vec{\nabla} f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$

EXAMPLE

Lets find the tangent approximation to the surface $z = f(x, y) = 1 - 2x^2 - y^2$ at $(1/2, 1/2)$.

EXERCISE

Williamson & Trotter, page 232, #12. Find the tangent approximation $T(\vec{x})$ to the function $f(x, y, z) = (x - y)z$ at $(1, 0, 1)$.

DEFINITION

For a function $f : \mathbb{R}^n \to \mathbb{R}$ and a direction $\vec{v}$ the **directional derivative** is defined as

$$\frac{\partial f}{\partial \vec{v}} = \lim_{t \to 0} \frac{f(\vec{x} + t\vec{v}) - f(\vec{x})}{t} = \vec{\nabla} f(\vec{x}) \cdot \vec{v}$$

NOTE: that if the direction $\vec{v} = \hat{e}_k$ then the directional derivative in that direction (i.e. parallel to the x_k axis) is simply the partial derivative with respect to x_k.

$$\frac{\partial f}{\partial \hat{e}_k} = \vec{\nabla} f(\vec{x}) \cdot \hat{e}_k = \frac{\partial f}{\partial x_k}$$

EXERCISE

Williamson & Trotter, page 262, #6. Find the directional derivative at $\vec{x} = (1, 0)$ in the direction $\vec{v} = (\cos(\alpha), \sin(\alpha))$ of $f(x, y) = e^x \sin(y)$.