Multivariable Calculus

Math 212 Spring 2006 Fowler 112 MWF 8:30pm - 9:25am
 http://faculty.oxy.edu/ron/math/212/06/

Class 12: **Friday February 17**

SUMMARY Limits of a Multivariable Function
CURRENT READING Williamson & Trotter, Section (Section 5.1)
HOMEWORK Williamson & Trotter, page 224: # **2, 3, 4, 5, 8**, 12, 25, 26, 27, 32
Extra Credit page 225: # 42

In order to begin our discussion of differentiability of a vector function of a vector input variable we will need to define a number of new terms.

DEFINITION: **neighborhood**

For a given value of $\delta > 0$, a δ-ball is the set of points $\vec{x}$ in $\mathbb{R}^n$ which satisfy the inequality that $|\vec{x} - \vec{x}_0| < \delta$. Another name for this set of points is **neighborhood**, which is sometimes denoted $N_\delta(\vec{x}_0)$.

DEFINITION: **limit point**

A **limit point** (sometimes called a cluster point) $\vec{x}$ of a set S is a point (not necessarily in S) for which *every* δ neighborhood of $\vec{x}$ contains at least one point which belongs to S.

DEFINITION: **interior point**

An **interior point** is a point $\vec{x}$ in a set S for which *there exists* a δ neighborhood of $\vec{x}$ which only contains points which belong to S.

DEFINITION: **boundary point**

A **boundary point** is a point $\vec{x}$ in a set S for which *every* δ neighborhood of $\vec{x}$ contains both a point which is in S and a point which is not in S.

DEFINITION: **open set**

An **open set** is a set S for which *every* element of S is an interior point.

DEFINITION: **closed set**

A **closed set** is a set S which contains *every* limit point of S. **Note:** by definition, every boundary point of a set is a limit point. So a closed set contains all of its boundary points.

EXAMPLE

Consider $D_1 = \{x : a \leq x \leq b\}$, $D_2 = \{\vec{x} : a \leq x_1 \leq b, c < x_2 < d\}$, $D_3 = \{\vec{x} : |\vec{x} - \vec{x}_0| < 1\}$, $D_4 = \mathbb{R}^3$ and $D_5 = \{\vec{x} : \vec{x} = (1,1)\}$. Describe whether these sets are open, closed, neither open nor closed or both open and closed.

DEFINITION: **limit**

Consider a function $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$. Let $\vec{y}_0$ be a point in $\mathbb{R}^m$ and $\vec{x}_0$ is a limit point of the domain of f in $\mathbb{R}^n$. We say that $\vec{y}_0$ is the **limit** of $\vec{f}$ at $\vec{x}_0$ if for a given ϵ there exists a $\delta > 0$ such that $|\vec{f}(\vec{x}) - \vec{y}_0| < \epsilon$ whenever $\vec{x}$ is in the domain of $\vec{f}$ and satisfies $0 < |\vec{x} - \vec{x}_0| < \delta$. This process is denoted by

$$\lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x}) = \vec{y}_0$$

EXAMPLE

Consider $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Show that $\lim_{\vec{x} \to \vec{0}} \vec{f}$ does not exist.

(a) Take the limit by approaching $(0, 0)$ along the x-axis. What value do you get?

(b) Take the limit by approaching $(0, 0)$ along the y-axis. What value do you get?

(c) Take the limit by approaching $(0, 0)$ along the line $y = \alpha x$. What value do you get?

DEFINITION: **continuity**

A function $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous at** $\vec{x}_0$ IF (1) $\vec{x}_0$ is in the domain of $\vec{f}$ AND (2) $\lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x}) = \vec{f}(\vec{x}_0)$

Basically this says, that when $\vec{x}$ is "close" to $\vec{x}_0$, then $\vec{f}(\vec{x})$ is close to $\vec{f}(\vec{x}_0)$. This is the basic conceptual idea of **continuity**.

Theorem

A vector function of a vector variable is continuous at a point if and only if all of its coordinate functions are continuous at that point.

Theorem

Given two continuous functions $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{g}: \mathbb{R}^m \to \mathbb{R}^p$, $\vec{g}(\vec{f}(\vec{x}))$ is also continuous wherever it is defined. In other words, composite functions of continuous functions are continuous wherever they are well defined.

EXERCISE

Williamson & Trotter, page 224, #25. Determine the points at which the function fails to have a limit. Take the domain of each coordinate function as large as possible. The domain of $\vec{f}$ is then the part common to the domain of all the cordinate functions. The function is $f(u, v) = \left(\dfrac{uv}{1 - u^2 - v^2}, \dfrac{1}{2 - u^2 - v^2}. \right)$

Is the function continuous at every point on this domain? Is this domain open or closed, neither or both?