Multivariable Calculus

Math 212 Spring 2006 | Fowler 112 MWF 8:30pm - 9:25am

 | http://faculty.oxy.edu/ron/math/212/06/

Class 8: **Wednesday February 8**

SUMMARY Vector Functions
CURRENT READING Williamson & Trotter, Section (Section 4.1 and 4.2)
HOMEWORK Williamson & Trotter, page 182: **2**, 3, 9, 13, 14, **17**

DEFINITION

A **vector** function of a **vector** variable $\vec{f}(\vec{x})$ with **domain** $D \subset \mathbb{R}^n$ and **range** $R \subset \mathbb{R}^m$ means that it has possible input values which form a subset of $\mathbb{R}^n$ and the set of possible output values are a subset of $\mathbb{R}^m$. Often the notation $f : D \to R$ or $f : \mathbb{R}^n \to \mathbb{R}^m$ is used.

In order to produce the output of this vector function of a vector requires m **coordinate functions** which are scalar functions of a vector variable: $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$

Matrix Representation

If f is a linear vector function then $f_i(x_1, x_2, x_3, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$ then it can be replaced by a simple matrix-vector product $\vec{f}(\vec{x}) = A\vec{x}$

EXAMPLE 1

The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $y_1(x_1, x_2) = -x_1 - x_2$ and $y_2(x_1, x_2) = x_1 + 2x_2$ has what matrix form?

Vector Functions of a Single Variable

Section 4.1 and 4.2 deal with the calculus of vector functions of a single variable which have the form $f : \mathbb{R} \to \mathbb{R}^n$. We've previously seen an example of such a function, $f : \mathbb{R} \to \mathbb{R}^3$

EXAMPLE 2

What kind of geometric object is the image of the function $\vec{x}(t) = (1 + 3t, -1 - t, -2 + t)$?

Derivative of a Vector Function of a Single Variable
Basically vector functions of a single variable should be treated as collections of real functions of a real variable.

EXAMPLE 3

Consider $f : \mathbb{R} \to \mathbb{R}^4$ where $\vec{f}(t) = \begin{bmatrix} t^2+1 \\ \sin(t) \\ e^{-t} \\ 4 \end{bmatrix}$. Compute $\vec{f}(0)$, $\dfrac{d\vec{f}}{dt}$ and $\displaystyle\int \vec{f}(t)dt$

DEFINITION

If a curve has a parametric representation $\vec{f}(t)$ such that the image of $\vec{f}$ has a derivative $\vec{f}'(t)$ which is continuous **and** never zero, then the curve is described as **smooth**.

EXERCISE 1

Show that the curve defined by $\vec{x}(t) = (\cos(t), \sin(t), t)$ is a **helix** and is a **smooth curve**.

Tangent Vector and **Tangent Line**
The tangent vector to a parametric curve $\vec{x}(t)$ is given by the derivative of $\vec{x}$ with respect to t and can be denoted $\dot{\mathbf{x}}$. Using this information, it's clear that the equation of a tangent line $\vec{t}(t)$ to a curve $\vec{x}(t)$ at the point $\mathbf{x}_0$ is given by $\vec{t}(t) = \dot{\mathbf{x}}t + \mathbf{x_0}$

EXERCISE 2

Find the equation of the tangent line to the helical curve $\vec{x}(t)$ from the previous example at the point $\mathbf{x_0} = (1, 0, 0)$.

DEFINITION

Length of a curve $l(\gamma) = \displaystyle\int_{t_0}^{t_1} |\dot{\mathbf{x}}|dt$ where γ is the parameterized curve between $t = t_0$ and $t = t_1$.

EXERCISE 3

Find an expression for the length of one revolution of a helix from $t = 0$ to $t = 2\pi$.