Class 7: **Monday February 6**

SUMMARY Review of Linear Systems, part 3: The Inverse Matrix and Determinants
CURRENT READING Williamson & Trotter, Section (Chapter 2)
HOMEWORK Williamson & Trotter, page 86: **5**, 8, 11, **26**, 29 page 98: # 1, 2, 3, **4**, 21
Extra Credit page 99: # 22, 40

The Determinant

The **determinant** of a (square) matrix is a real number associated with that matrix. It is denoted by $\det(A)$ or sometimes $|A|$ and should be considered a function which has a square $n \times n$ matrix as its input and a real number as its output.

The **significance** of the determinant is that $\det(A) = 0 \Leftrightarrow$ **the matrix is singular**. Thus if you want to determine whether a linear system $A\vec{x} = \vec{b}$ has a unique solution all you need to do is find out whether the determinant of the coefficient matrix A is non-zero.

Unfortunately, computing the determinant of a matrix can be complicated. The most common method is to use the **Co-Factor Method**.

DEFINITION

Let A be any matrix. The ij**-minor** of A is the matrix obtained by removing its ith row and its jth column. It is denoted by $\hat{A}_{i,j}$. $\det(A) = \sum_{j=1}^{n} A_{i,j}C_{i,j}$ where $C_{i,j} = (-1)^{i+j}\det(\hat{A}_{i,j})$.
The coefficient $C_{i,j}$ defined above is called the **cofactor** of the entry $A_{i,j}$.

EXERCISE 1

Compute the determinant of $\begin{bmatrix} 2 & 6 & 2 \\ 0 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}$

Properties of the Determinant

1. An elementary multiplication of a row of A by a scalar r gives $r\det(A)$
2. An elementary modification of A leaves $\det(A)$ unchanged.
3. Interchanging two rows of A changes the sign of $\det(A)$.

Cramer's Rule
Given $A\vec{x} = \vec{b}$, the jth coordinate of $\vec{x}$ is given by the formula

$$x_j = \frac{\det(B_j)}{\det(A)}$$

where B_j is obtained by replacing the j-th column of A by $\vec{b}$.

EXAMPLE 1 Solve the system

$$\begin{aligned} 2x + 4y &= 1 \\ x + 3y &= 2 \end{aligned}$$

The Inverse Matrix

Given a square matrix A if there exist a square matrix B such that $AB = BA = I$ then the matrix B is called the inverse matrix and is denoted A^{-1}. $\det(A) \neq 0 \Leftrightarrow A^{-1}$ **exists**. When A^{-1} exists the matrix is called **invertible**.

The usefulness of knowing the inverse of a matrix is that it help you solve $A\vec{x} = \vec{b}$ by direct computation i.e. $\vec{x} = A^{-1}\vec{b}$. If A is invertible, then the only solution to the homogeneous system $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

Inverse of a Matrix Product
$(AB)^{-1} = B^{-1}A{-}1$ and $(ABC)^{-1} = C^{-1}B^{-1}A{-}1$

Gauss-Jordan Elimination
One way to compute the inverse of a matrix A is to apply Gaussian elimination to the augmented matrix consisting of $[A|I]$

EXAMPLE 2

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 6 & 4 & -3 & 0 & 1 & 0 \\ 9 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

EXERCISE 2

Use your knowledge of inverses to confirm that the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$