Multivariable Calculus

Math 212 Spring 2006 — Fowler 112 MWF 8:30pm - 9:25am

 — http://faculty.oxy.edu/ron/math/212/06/

Class 4: **Monday January 31**

SUMMARY Euclidean Geometry and The Vector Cross Product
CURRENT READING Williamson & Trotter, Section 1.5 and Section 1.6
HOMEWORK Williamson & Trotter, Page 36 # 1,3,**9**,11,15,18,**22** ; Page 42-44 # 1, 2, 3, 4, 8, **9, 11, 22**; EXTRA CREDIT page 42 #15.

General Equation of a Plane in Euclidean Space

The main way we often think of planes in euclidean space (i.e. the space we are used to living in where lines are perfectly "straight" and go on forever) is to define a plane in $\mathbb{R}^3$ which contains the point $\vec{x}_0$ as that 2-D object which is exactly perpendicular to a particular vector $\vec{p}$:

$$\vec{p} \cdot (\vec{x} - \vec{x}_0) = 0$$

Note that in $\mathbb{R}^2$ you can define the equation of a line this way, also.
Another way to write this is to think of the vector emanating from the plane normal (i.e. perpendicularly) to the plane as a unit vector $\hat{n} = \vec{p}/|\vec{p}|$ so that

$$\hat{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

or

$$\hat{n} \cdot \vec{x} - c = 0$$

Note, $\vec{x}_0$ is a fixed (given) point on the plane. This form of the equation of a plane means that when you see the other standard form $Ax + By + Cz = D$ then you know the vector $\vec{p} = A\hat{i} + B\hat{j} + C\hat{k}$ is perpendicular to the plane and that $D = Ax_0 + By_0 + Cz_0$ and the point (x_0, y_0, z_0) is on the plane.

In fact, two planes are considered parallel to each other if their normal vectors $\hat{n}_1$ and $\hat{n}_2$ are parallel.

EXERCISE 1

Consider $2x + 3y - 4z = 6$ and $4x + 6y - 8z = 9$. Are these two objects lines or planes? Are they parallel to each other? Write down an example of a vector which is perpendicular to each one of these objects.

Distance from a Point to a Plane

The distance δ from a point in space $\vec{x}_1$ to a plane (or line) $\hat{n}\cdot(\vec{x}-\vec{x_0})=0$ or $\hat{n}\cdot\vec{x}-c=0$ is given by

$$\delta=\hat{n}\cdot(\vec{x}_1-\vec{x}_0)=\hat{n}\cdot\vec{x}_1-c=0$$

EXERCISE 2

What is the distance between the point **P** $(1,1,1)$ and the plane $x+y-2z=-1$? Which point is closer to the plane, **P** or the origin?

The Vector Cross Product

DEFINITION

The **cross product** of two vectors $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ in $\mathbb{R}^3$ is defined to be the vector

$$\vec{u}\times\vec{v}=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$$

Luckily, there's an easy way to remember this calculation as the determinant of a 3x3 matrix

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=\begin{vmatrix}u_2&u_3\\v_2&v_3\end{vmatrix}\hat{i}-\begin{vmatrix}u_1&u_3\\v_1&v_3\end{vmatrix}\hat{j}+\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}\hat{k}$$

EXAMPLE

Find the cross product $\vec{u}\times\vec{v}$ of $\vec{u}=(1,-3,2)$ and $\vec{v}=(2,4,-5)$.

Take the dot product of your answer with both $\vec{u}$ and $\vec{v}$. What do you notice?

Properties of The Cross Product

$$\vec{u}\cdot(\vec{u}\times\vec{v})=0$$

$$\vec{v}\cdot(\vec{u}\times\vec{v})=0$$

$$\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$$

$$\hat{\imath}\times\hat{\jmath}=k;\hat{\jmath}\times\hat{k}=\hat{\imath};\hat{k}\times\hat{\jmath}=\hat{\imath}$$

Additivity: $\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$ and $(\vec{u}+\vec{v})\times\vec{w}=\vec{u}\times\vec{w}+\vec{v}\times\vec{w}$
Homogeneity: $r(\vec{u}\times\vec{v})=(r\vec{u})\times\vec{v}=\vec{u}\times(r\vec{v})$
Area of a Parallelogram $=|\vec{u}\times\vec{v}|$
Volume of a Parallelpiped $=\vec{u}\cdot(\vec{v}\times\vec{w})$