Test 2: Multivariable Calculus

Math 224
Ron Buckmire

Friday April 16 2004
2:30pm-3:30pm

Name: _______________  MB = ___________

Directions: Read all problems first before answering any of them. Questions 1 and 2 are related. This is a one hour, open-notes, open book, test. No calculators. You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your “scratch work.”

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1
1. (20 points) Chain Rule, Implicit Function Theorem.
Consider a surface implicitly-defined as \( F(x, y, z) = 0 \) which can be written as \( z = f(x, y) \) so that \( F(x, y, f(x, y)) = 0 \).

a. (10 points) Use the Chain Rule to show that
\[
\frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial F}{\partial z} = 0
\]

\[
\frac{\partial x}{\partial x} = 1 \quad \frac{\partial y}{\partial y} = 0
\]

\[
F_x + F_z \frac{\partial z}{\partial x} = 0
\]

\[
\frac{\partial F}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0
\]

\[
F_y + F_z \frac{\partial z}{\partial y} = 0
\]

b. (10 points) Use the implicit function theorem to obtain the equivalent result,
that is,
\[
\frac{\partial z}{\partial x} = -\frac{\partial x}{\partial F} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial y}{\partial F}
\]

\[
\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad \mathbf{G}(\mathbf{x})
\]

\[
\mathbf{G}'(\mathbf{x}) = -\left(\mathbf{F}_y\right)^{-1} \mathbf{F}_x
\]

In this case \( \mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) \) so \( \mathbf{x} = (x, y) \)

\[
\mathbf{F} = \mathbf{F} \quad \mathbf{G} = \mathbf{f} \quad \mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^1
\]

\[
\mathbf{G}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = -\left(\frac{\partial F}{\partial z}\right)^{-1} \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix}
\]

\[
\mathbf{F}_x = \frac{\partial F}{\partial z}
\]

\[
\mathbf{F}_y = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix}
\]
2. **(20 points)** Partial Differentiation, Gradient Operator.
Consider \(\cos(x + y + z) = xyz\) as an example of \(F(x, y, z) = 0\) and \(z = f(x, y)\) from Question 1.

**a.** *(10 points)* Write down \(F\) and \(f\), if you can. (If you can’t, explain why the requested expression can not be obtained.) How are these two functions different?

\[
F(x, y, z) = \cos(x + y + z) - xyz = 0
\]

\(z = f(x, y)\) **is IMPLICITLY DEFINED** and can **not be written down as an explicit expression**

\(F\) **is explicit**.

**b.** *(10 points)* Write down \(\nabla F\) and \(\nabla f\), if you can. (If you can’t, explain why the requested expression can not be obtained.) How are these two **vectors** different?

\[
\nabla F = (F_x, F_y, F_z) \quad \text{in } \mathbb{R}^3
\]

\[
= (-\sin(x + y + z) - yz, -\sin(x + y + z) - xz, -\sin(x + y + z) - xy)
\]

\[
\nabla f = (f_x, f_y) = \left(\frac{-F_x}{F_z}, \frac{-F_y}{F_z}\right) \quad \text{in } \mathbb{R}^2
\]

\[
= \left(\frac{+\sin(x + y + z) + yz}{-\sin(x + y + z) - xy}, \frac{-\sin(x + y + z) - xy}{+\sin(x + y + z) - xy}\right)
\]
3. (20 points) Iterated Integration.

a. (10 points) Evaluate \( \int_{-3}^{0} \int_{0}^{2} \sin(x+y+z) - \frac{x^2yz}{2} \, dz \, dy \)

\[
\int_{-3}^{0} \int_{0}^{2} \sin(x+y+z) - \frac{x^2yz}{2} \, dz \, dy = \int_{-3}^{0} \int_{0}^{2} \sin(1+y+z) - \sin(-1+y+z) \, dz \, dy
\]
\[
= \int_{-3}^{0} \left[ -\cos(1+y+z) + \cos(-1+y+z) \right]_{0}^{2} \, dy
\]
\[
= \int_{-3}^{0} \left[ -\cos(3+y) + \cos(1+y) + \cos(1+y) - \cos(-1+y) \right] \, dy
\]
\[
= \int_{-3}^{0} \left[ -\sin(3+y) + 2\sin(1+y) + \sin(-1+y) \right] \, dy
\]
\[
= \left[ -\sin(3+y) + 2\sin(1+y) - \sin(-1+y) \right]_{-3}^{0}
\]
\[
= -\sin(3) + 2\sin(1) - \sin(-1) - \left( -\sin(0) + 2\sin(-2) - \sin(-4) \right)
\]
\[
= -\sin(3) + 2\sin(1) + \sin(1) - 2\sin(-2) + \sin(-4)
\]
\[
= 3\sin(1) + 2\sin(2) - \sin(3) - \sin(4)
\]

b. (10 points) Evaluate \( \int_{1}^{2} \int_{0}^{\ln x} \frac{1}{x} \, dy \, dx \)

\[
\int_{1}^{2} \int_{0}^{\ln x} \frac{1}{x} \, dy \, dx = \int_{0}^{\ln 2} \int_{e^y}^{1} \frac{1}{x} \, dx \, dy
\]
\[
= \int_{0}^{\ln 2} \frac{1}{x} \, dx \, dy
\]
\[
= \ln x \bigg|_{e^y}^{1} \, dy
\]
\[
= \ln 2 - \ln(e^y) \, dy
\]
\[
= \ln 2 - y \, dy
\]
\[
= (\ln 2)^2 - \left( \frac{\ln 2}{2} \right)^2
\]
\[
= \frac{(\ln 2)^2}{2} - \frac{(\ln 2)^2}{4}
\]
\[
= \frac{(\ln 2)^2}{4}
\]
4. (20 points) Multiple Integration.

a. (10 points) Evaluate $\int_0^\pi \int_{-2}^2 ye^{r \cos \theta} r \, dr \, d\theta$ where $R$ is the first quadrant of the circle of radius 4 centered at the origin. (Sketch the region $R$).

\[
\int_0^\pi \int_{-2}^2 ye^{r \cos \theta} r \, dr \, d\theta = \int_0^\pi \int_{-2}^2 r^2 e^{r \cos \theta} \, dr \, d\theta
\]

\[
= -\frac{1}{2} \int_0^4 e^r + re^{-r} \bigg|_0^4
\]

\[
= -8 + 4e^4 - e^4 - (0 - e^0)
\]

\[
= -7 + 3e^4
\]

b. (10 points) Consider $\int_0^1 \int_0^{1-y} \int_0^{1-y} \, dz \, dy \, dx = \frac{1}{12}$. Re-compute this integral using a different triple integral which represents the same volume.

\[
\int_0^1 \int_0^{1-y} \int_0^{1-y} \, dz \, dx \, dy = \int_0^1 [1-y] \, dx \, dy
\]

\[
= \int_0^1 y^2 (1-y) \, dy = \int_0^1 y^3 + y^2 \, dy = \frac{y^4}{4} + \frac{y^3}{2} \bigg|_0^1
\]

\[
= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}
\]
5. (20 points) Constrained Multivariable Optimization, Lagrange Multipliers

The “geometric mean” of \( n \) numbers is defined as \( f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 x_3 \cdots x_n} \). Suppose that \( x_1, x_2, \ldots, x_n \) are positive numbers such that \( \sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n = c \), where \( c \) is a constant.

(a) (10 points) Find the maximum value of the geometric mean of \( n \) positive numbers given the constraint that their sum must be equal to a constant. [HINT: Consider \( f^n \) instead of \( f! \)]

\[
\begin{align*}
\phi &= f^n = x_1 x_2 x_3 x_4 \cdots x_n \\
g &= x_1 + x_2 + x_3 + \cdots + x_n \\
\nabla \phi &= \nabla (f^n) = \lambda \nabla g \\
\nabla f^n &= (x_1 x_3, x_2, x_4, \ldots, x_n, x_1, x_3, x_4, \ldots, x_n) \\
\nabla g &= (1, 1, 1, \ldots, 1) \\

x_1 x_3 &= \lambda x_1 \\
x_2 &= \lambda x_2 \\
x_4 &= \lambda x_4 \\
\vdots \\
x_n &= \lambda x_n \\
X_1 + X_2 + \cdots + X_n &= c \\

\phi &= \sqrt[n]{\lambda^c} \\
\lambda &= \frac{c^n}{n} \\
\text{MAX} &= \frac{c^n}{n} \\

\end{align*}
\]

(b) (10 points) You can deduce from part (a) that the geometric mean of \( n \) numbers is always less than or equal to the arithmetic mean, that is:

\[
\sqrt[n]{x_1 x_2 x_3 \cdots x_n} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}
\]

Under what conditions will the geometric mean be exactly equal to the arithmetic mean of those same \( n \) numbers?

When \( x_1 = x_2 = x_3 = \ldots = x_n = x \)

geometric mean = \( \sqrt[n]{x^n} = x \)

arithmetic mean = \( \frac{nx}{n} = x \)

When \( \sum_{i=1}^{n} x_i = c \) the max value of \( \sqrt[n]{\sum_{i=1}^{n} x_i} = c/n \)

So \( \sqrt[n]{\sum_{i=1}^{n} x_i} \leq \frac{c}{n} \)
EXTRA CREDIT (10 points) Unconstrained Multivariable Optimization

Consider \( f(x, y) = x^4 + y^4 - 4xy + 1 \).

a. (5 points) Find the three critical points of \( f(x, y) \).

Critical Points occur at \( \nabla^2 f = 0 \)

\[
\begin{align*}
\frac{\partial}{\partial x} f &= 4x^3 - 4y = 0 \\
\frac{\partial}{\partial y} f &= 4y^3 - 4x = 0
\end{align*}
\]

\( y^9 - y = 0 \)

\( (y^8 - 1)y = 0 \)

\( (y^4 - 1)(y^4 + 1)y = 0 \)

\( (y^2 - 1)(y^2 + 1)(y^4 + 1)y = 0 \)

\( y = 0 \) \( y = 1, y = -1 \)

\( x = -y^3 \) \( x = 1, x = -1 \)

\( 0, 0, 1 \)

\( 1, 1, -1 \)

\( -1, -1, -1 \)

b. (5 points) Use the Second Derivative Test to classify each of the three critical points of \( f(x, y) \).

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} f &= 12x^2 \\
\frac{\partial^2}{\partial y^2} f &= 12y^2 \\
\frac{\partial^2}{\partial x \partial y} f &= -4
\end{align*}
\]

At \((0, 0, 1)\)

\( D = 0 \cdot 0 - (-4)^2 = -16 < 0 \) \( \Rightarrow \text{Saddle} \)

At \((1, 1, -1)\)

\( D = 12 \cdot 12 - (-4)^2 = 144 - 16 = 128 > 0 \) \( \text{Local Min} \)

\( \frac{\partial^2}{\partial x^2} f > 0 \) \( \frac{\partial^2}{\partial y^2} f > 0 \)

At \((-1, -1, -1)\)

\( D = 12 \cdot 12 - (-4)^2 = 144 - 16 > 0 \) \( \Rightarrow \text{Local Min} \)

\( \frac{\partial^2}{\partial x^2} f > 0 \) \( \frac{\partial^2}{\partial y^2} f > 0 \)