Test 1: Multivariable Calculus

Math 212 — Monday February 27 2006
Ron Buckmire — 8:30pm-9:25am

Name: Key

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a one hour, open-notes, open book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

No.	Score	Maximum
1		30
2		40
3		30
BONUS		5
Total		**100**

1. Multivariable Functions, Differentiation, Point Sets. *30 points.*

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once. .

(a) *10 points.* **TRUE or FALSE?** "There does not exist a vector function of a scalar variable whose derivative equals itself."

$\boxed{\text{FALSE}}$

$$\vec{x}(t) = \begin{pmatrix} e^t \\ e^t \\ \vdots \\ e^t \end{pmatrix} \qquad \frac{d\vec{x}}{dt} = \begin{pmatrix} e^t \\ e^t \\ \vdots \\ e^t \end{pmatrix} = \vec{x}$$

There does exist a vector function $\vec{x}(t)$ such that $\vec{x}'(t) = \vec{x}(t)$ for all t.

(Another answer is $\vec{x}(t) = \vec{0}$ $\quad \vec{x}'(t) = \vec{0} = \vec{x}(t)$ but that one's boring!)

(b) *10 points.* **TRUE or FALSE?** "There does not exist a scalar function of a vector variable whose gradient equals itself."

$\boxed{\text{TRUE}}$

The gradient of a scalar function of a vector variable is a vector function!

Given $f(\vec{x})$ where $\vec{x} \in \mathbb{R}^n$, $\vec{\nabla} f(\vec{x}) \in \mathbb{R}^n$

so $f(\vec{x}) \neq \vec{\nabla} f(\vec{x})$

(c) *10 points.* **TRUE or FALSE?** "There does not exist a point set in $\mathbb{R}^n$ which is both open and closed."

$\boxed{\text{FALSE}}$

$\mathbb{R}^n$ is both open and closed.

It contains all its interior points

It does not have any boundary points, so it does include them all.

3. Partial Derivatives. *30 points.*

The energy E, of a body of mass m moving with speed $v \geq 0$ is given by the formula

$$E = mc^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

where c is the speed of light.

(a) (*15 points.*) Compute $\frac{\partial E}{\partial m}$. What do you expect the sign of $\frac{\partial E}{\partial m}$ to be? **EXPLAIN YOUR ANSWER.** (Write down a few sentences explaining the physical interpretation of your answer.)

$$\frac{\partial E}{\partial m} = \frac{c^2}{\sqrt{1-\frac{v^2}{c^2}}} - c^2$$

$$= c^2 \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \right) > 0$$

$v < c \Rightarrow \frac{v}{c} < 1 \Rightarrow \frac{v^2}{c^2} < 1$

$1 - \frac{v^2}{c^2} > 0$

$1 - \frac{v^2}{c^2} < 1$

$\sqrt{1 - \frac{v^2}{c^2}} < 1$

$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} > 1$

$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 > 0$

The rate of change of Energy as mass increases with speed constant also increases. $m\uparrow, E\uparrow$ and $m\downarrow E\downarrow$

(b) (*15 points.*) Compute $\frac{\partial E}{\partial v}$ What do you expect the sign of $\frac{\partial E}{\partial v}$ to be? **EXPLAIN YOUR ANSWER.** (Write down a few sentences explaining the physical interpretation of your answer.)

$$\frac{\partial E}{\partial v} = mc^2 \cdot -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot -\frac{1}{c^2} \cdot 2v - 0$$

$$= + \frac{mv}{\sqrt{\left(1-\frac{v^2}{c^2}\right)^3}} > 0$$

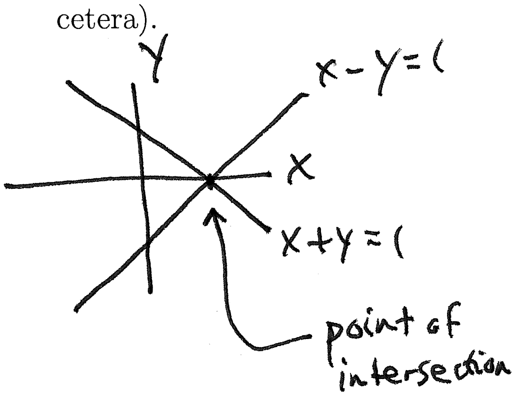
The energy increases as mass is held constant but speed increases. $v\uparrow, E\uparrow$ and $v\downarrow, E\downarrow$

2. Analytic Geometry in $\mathbb{R}^n$, Lines, Planes. Recall that a hyperplane in $\mathbb{R}^n$ is represented by an equation of the form $\sum_{k=1}^{n} c_k x_k = c_0$. This problem is about the geometric objects created by the intersection of two hyperplanes in $\mathbb{R}^n$.

(a) (*10 points.*) Consider the following two hyperplanes in $\mathbb{R}^2$:

$$\begin{aligned} x + y &= 1 \\ x - y &= 1 \end{aligned}$$

Do there exist points in $\mathbb{R}^2$ that lie on both of these objects simultaneously? If so, find all such points of intersection (i.e. a vector equation in $\mathbb{R}^2$ describing the object which is made up of these special points) and describe the object (i.e. name it) and give the dimension of the object (i.e. zero-dimensional, one-dimensional, two-dimensional, three-dimensional, et cetera).



Yes, there is a single point of intersection of two non parallel lines in $\mathbb{R}^2$

$2x = 2 \Rightarrow x = 1$

If $x = 1$, $y = 0$

$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

This is a zero dimensional object.

(b) (*10 points.*) Consider the following two hyperplanes in $\mathbb{R}^3$:

$$\begin{aligned} x + y + z &= 1 \\ x + y - z &= 1 \end{aligned}$$

Do there exist points in $\mathbb{R}^3$ that lie on both of these objects simultaneously? If so, find all such points of intersection (i.e. a vector equation in $\mathbb{R}^3$ describing the object which is made up of these special points) and describe the object (i.e. name it) and give the dimension of the object (i.e. zero-dimensional, one-dimensional, two-dimensional, three-dimensional, et cetera).

Two nonparallel planes intersect at $\boxed{\text{a line}}$ in $\mathbb{R}^3$

$x + y = 1$
$z = 0$

$\vec{x} = \begin{pmatrix} 1-y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Lines are ONE DIMENSIONAL OBJECTS.

(c) (*10 points.*) Consider the following two hyperplanes in $\mathbb{R}^4$:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Do there exist points in $\mathbb{R}^4$ that lie on both of these objects simultaneously? If so, find all such points of intersection (i.e. a vector equation in $\mathbb{R}^4$ describing the object which is made up of these special points) and describe the object (i.e. name it) and give the dimension of the object (i.e. zero-dimensional, one-dimensional, two-dimensional, three-dimensional, et cetera).

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_4 &= 0 \end{aligned}$$

Two non-parallel hyperplanes in $\mathbb{R}^4$ must intersect at a plane in $\mathbb{R}^4$. Planes are TWO DIMENSIONAL OBJECTS

$$\vec{x} = \begin{pmatrix} 1 - x_2 - x_3 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \underbrace{\vec{p} + x_2\vec{u} + x_3\vec{v}}_{\text{vector eq}^{\text{n}}\text{ of plane}}$$

(d) (*10 points.*) **(The point!)** Consider the following two hyperplanes in $\mathbb{R}^n$:

$$\begin{aligned} x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n &= 1 \\ x_1 + x_2 + x_3 + \ldots + x_{n-1} - x_n &= 1 \end{aligned}$$

Do there exist points in $\mathbb{R}^n$ that lie on both of these objects simultaneously? If so, find all such points of intersection (i.e. a vector equation in $\mathbb{R}^n$ describing the object which is made up of these special points) and describe the object (i.e. name it) and give the dimension of the object (i.e. zero-dimensional, one-dimensional, two-dimensional, three-dimensional, et cetera).

Two non-parallel hyperplanes in $\mathbb{R}^n$ will intersect to produce an $n-2$ dimensional object, called a "$n-2$ plane."

Uses variables from x_2 to x_{n-1} means $n-2$ vectors = dimension

$$\begin{aligned} (x_1 \text{ is missing}) \quad x_1 + x_2 + x_3 + x_4 + \ldots x_{n-1} &= 1 \\ x_n &= 0 \end{aligned}$$

$$\vec{x} = \begin{pmatrix} 1 - x_2 - x_3 - x_4 - \ldots - x_{n-1} \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} \cancel{0}-1 \\ +1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots x_{n-1} \begin{pmatrix} -1 \\ \\ \\ \\ 1 \\ 0 \end{pmatrix}$$

BONUS QUESTION. Vectors, Dot Product, Magnitude. (*5 points.*)
Consider the following statements.

Let $\vec{u}$ be a vector such that $|\vec{u}| = 1$. Choose a vector $\vec{v}$ such that $\vec{u} \cdot \vec{v} = 3$ and $|\vec{v}| = \sqrt{5}$.

$$\begin{aligned} |\vec{u} - \vec{v}|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\ &= 1 - 2(3) + 5 \\ &= 0 \end{aligned}$$

This implies that $\vec{u} = \vec{v}$ since $\vec{u} - \vec{v} = \vec{0}$. But $\vec{u}$ and $\vec{v}$ have different lengths.

HOW CAN TWO VECTORS BE EQUAL YET HAVE DIFFERENT LENGTHS? FIND THE ERROR! EXPLAIN YOUR ANSWER THOROUGHLY, EXTRA CREDIT POINTS ARE HARD TO GET.

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta_{u,v}$

If $|\vec{u}| = 1$ and $|\vec{v}| = \sqrt{5}$ and $\vec{u} \cdot \vec{v} = 3$

$3 = 1 \cdot \sqrt{5} \cos \theta_{u,v}$

$\frac{3}{\sqrt{5}} = \cos \theta_{u,v}$ and $\frac{3}{\sqrt{5}} > 1$

But $\sqrt{5} \cancel{<} 3$! So $\frac{\sqrt{5}}{3} \cancel{<} 1$ so $\cos \theta_{u,v} > 1$ which is impossible.
$(5 < 9)$

There does not exist a pair of vectors such that $|\vec{u}| = 1$ $|\vec{v}| = \sqrt{5}$ and $\vec{u} \cdot \vec{v} = 3$.