Multivariable Calculus

Math 212 Fall 2005 — Fowler 307 MWF 9:30pm - 10:25am

 — http://faculty.oxy.edu/ron/math/212/05/

Class 29: **Monday December 5**

SUMMARY Green's Theorem
CURRENT READING Williamson & Trotter, §9.1
HOMEWORK page 408: 3, 4, 6, 7, 10 **Extra Credit page 409: 15, 18, 20**

THEOREM: Green's Theorem

Given a planar region D whose boundary is a single closed curve γ parametrized by a function $\vec{g}(t)$ so that as t increases from a to b, $\vec{g}(t)$ traces out γ once in the *counter-clockwise* direction, then if $F(x, y)$ and $G(x, y)$ are real-valued functions defined on D including its boundary, then the formula for Green' Theorem is:

$$\int\int_D \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dx\ dy = \oint_\gamma F\ dx + G\ dy \qquad \textbf{(Green's Theorem)}$$

EXAMPLE 1

Williamson & Trotter, page 408, #1. Use Green's Theorem to compute the value of the line integral $\oint_\gamma y\ dx + x^2\ dy$ where γ is the indicated path.
γ is the circle given by $g(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Exercise 1

Williamson & Trotter, page 408, #2. Repeat problem #1, this time with γ being the square with corners at $(\pm 1, \pm 1)$, traced counter-clockwise.

EXAMPLE 2

Williamson & Trotter, page 408, #11. Show that if D is a simple region bounded by a piecewise smooth curve γ, traced counter clockwise, then the area of the interior of γ (i.e. the area of D) is given by

$$A(D) = \frac{1}{2}\oint_\gamma -y\ dx + x\ dy$$

Now that we know about **curl** and **div** we can use them to re-write the Green's Theorem result. Note that the integrand in the area integral in Green's Theorem is the non-zero component in $\vec{\nabla} \times \vec{F}$ where $\vec{F}(x,y) = (F(x,y), G(x,y), 0)$

THEOREM: Stokes' Theorem in the Plane

So, in vector format, we can write Green's Theorem as

$$\int\int_D (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\ dA = \oint_{\partial D} \vec{F} \cdot d\vec{x}$$

This result is known as **Stokes' Theorem in the Plane**. Note that $d\vec{x} = \hat{t}\left|\frac{d\vec{g}}{dt}\right| ds$ where s is a parameter representing arc length along the curve γ (not important) inn a direction $\hat{t}$ always tangent to the curve.

In Stokes' Theorem we are dealing with a vector field in $\mathbb{R}^2$ which has the form $\vec{F} = (F(x,y), G(x,y))$. In Gauss' Theorem we are dealing with a vector field which has the form $\vec{H} = (-G(x,y), F(x,y))$

Suppose we had a unit vector $\hat{n}$ which is orthogonal in the plane (i.e. normal) to the curve γ at every point. Then $\hat{n} \cdot \hat{t} = 0$ and $\vec{F} \cdot d\vec{x} = (F, G) \cdot (dx, dy) = F\ dx + G\ dy = -G\ (-dy) + F\ dx = (-G, F) \cdot (-dy, dx) = \vec{H} \cdot \hat{n}\ ds$

This also means that $\vec{H} \cdot d\vec{x} = \vec{F} \cdot \hat{n} ds$.

Since $\vec{H} \cdot d\vec{x} = (-G, F) \cdot (dx, dy) = -G\ dx + F\ dy = (F, G) \cdot (dy, -dx) = \vec{F} \cdot \hat{n}\ ds$

But using Green's Theorem on $\vec{H} = (-G(x,y), F(x,y))$ and recalling $\vec{F} = (F(x,y), G(x,y))$ produces

$$\begin{aligned}\int_\gamma \vec{H} \cdot d\vec{x} = \int_\gamma -G(x,y)\ dx + F(x,y)\ dy &= \int\int_D \left(\frac{\partial}{\partial x}[F(x,y)] - \frac{\partial}{\partial y}[-G(x,y)]\right)\ dA \\ \int_\gamma \vec{F} \cdot \hat{n}\ ds &= \int\int_D \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right)\ dA\end{aligned}$$

But the right hand side of this expression should remind you of the divergence operator. So we can also re-write Green's theorem using another differential vector operator, this time **div** $\vec{F}$

THEOREM: Gauss' Theorem in the Plane

$$\int\int_D \vec{\nabla} \cdot \vec{F}\ dA = \int_{\partial D} \vec{F} \cdot \hat{n}\ ds$$

EXAMPLE 1

Willliamson & Trotter, Page 408, # 9. Evaluate $\int_C (x^2 - y^2)\ dx + (x^2 + y^2)\ dy$ where C is the circle of radius 1 centered at the origin and traced clockwise.

Exercise 1

Willliamson & Trotter, Page 408, # 12. Let f be a real valued function with continuous second order derivatives in an open set D in $\mathbb{R}^2$. Let $\vec{F}$ a vector field defined in D by $\vec{F} = \vec{\nabla} f(\vec{x})$. Show that if $\vec{F} = (F(\vec{x}), G(\vec{x}))$, then the equation $\dfrac{\partial G}{\partial x} - \dfrac{\partial F}{\partial y} = 0$ at all points in region D.