Multivariable Calculus

Math 212 Fall 2005 | Fowler 307 MWF 9:30pm - 10:25am

 | http://faculty.oxy.edu/ron/math/212/05/

Class 25: **Monday November 14**

SUMMARY Change of Variables Theorem
CURRENT READING Williamson & Trotter, Section 7.4
HOMEWORK #24 Williamson & Trotter, page 346: 3, 6, 8, 9, 17, 18, 19 **Extra Credit page 347: 22, 26**

Suppose we want to change variables from an integral defined over $T(R)$ over one that is defined over R where $\vec{x} = \vec{T}(\vec{u}) : \mathbb{R}^n \to \mathbb{R}^n$

Jacobi's Theorem

Given $\vec{T} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable transformation and R is a subset of $\mathbb{R}^n$ having a boundary consisting of finitely many smooth sets. IF R and its boundary are contained in the domain of $\vec{T}$ and that **(i)** $\vec{T}$ is one-to-one on the interior of R and **(ii)** det $(\vec{T}_{\vec{u}}) \neq 0$ in the interior of R, THEN

$$\int_{T(R)} f(\vec{x})\, dV_x = \int_R f(\vec{T}(\vec{u}))\, |\det(\vec{T}_{\vec{u}}(\vec{u}))|\, dV_u$$

or, using Leibnizian Notation where T maps from W^* in uvw-space to $W = T(W^*)$ in xyz-space

$$\int\int\int_W f(x,y,z)\, dx\, dy\, dz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw \text{ in } \mathbb{R}^3$$

and

$$\int\int_W f(x,y)\, dx\, dy = \int\int_{W^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \text{ in } \mathbb{R}^2$$

Generally, we use this theorem to convert from Cartesian coordinates to polar, spherical, and cylindrical co-ordinates.

Change of Variables: Polar Coordinates

$$\int\int_D f(x,y)\, dx\, dy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Change of Variables: Cylindrical Coordinates

$$\int\int_W f(x,y,z)\, dx\, dy\, dz = \int\int\int_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\, drd\theta dz$$

Change of Variables: Spherical Coordinates

$$\int\int\int_W f(x,y,z)\, dx\, dy\, dz = \int\int\int_{W^*} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2\sin\phi\, drd\theta d\phi$$

EXAMPLE 1

Williamson & Trotter, page 346, #7. Compute $\int_D \cos(x^2 + y^2)\ dx\ dy$ where D is the disk of radius $\sqrt{\pi/2}$ centered at (0,0).

Exercise 1

Williamson & Trotter, page 346, #12. Compute $\int_C z^2\ dx\ dy\ dz$ where C is the region in $\mathbb{R}^3$ described by $1 \leq x^2 + y^2 + z^2 \leq 4$

$\boxed{\text{PAIRED GROUPWORK}}$

Williamson & Trotter, page 364, #24. (a) Sketch the region R bounded by the graphs of $y = x^3$ and $x = y^2$. (b) The double integral $\int_R x \, dx \, dy$ is equal to each of two iterated integrals over R. Write down **both** of them. Each member of the pair evaluates a different integral and show they have the same value.

$\boxed{\textbf{Exercise 2}}$

Williamson & Trotter, page 365, #46.. The region of integration R for $h(x, y, z)$ is bounded above by $z = 2$ and below by the circular parabaloid $z = x^2 + y^2$; find the limits, which may be non-constant, for $\int_R h \, dV = \int_a^b \int_c^d \int_e^f h(x, y, z) \, dz \, dx \, dy$, the integral of h over the region R.

$\boxed{\text{EXAMPLE 2}}$

Williamson & Trotter, page 366, #61. $\displaystyle\int_0^{2\pi} d\theta \int_0^1 dr \int_0^{\sqrt{1-r^2}} r\, dz$

(a) Sketch the region of integration. **(b)** Write the integral in terms of rectangular coordinates. **(c)** Write the integral in terms of spherical coordinates. **(d)** Evaluate the integral.

$\boxed{\textbf{Exercise 3}}$

Williamson & Trotter, page 365, #41.. Let C be a solid cylinder of radius 1 symmetric about the z-axis. Let W be the wedge-shaped subset of C where $0 \leq z \leq x$. Write an iterated integral for $\displaystyle\int_W z\, dV$ **(a)** in rectangular coordinates. **(b)** in cylindrical coordinates. **(c)** Evaluate the multiple integral whichever way you prefer.