Class 22: **Monday November 7**

SUMMARY Curvilinear Coordinates
CURRENT READING Williamson & Trotter, Section 6.5
HOMEWORK #21 Williamson & Trotter, page 308:3,4,6,7,9,10,12; page 310: 10,21,29,30,36,37
Extra Credit page 309: 15; page 310: 16, 33, 38

Polar Coordinates

Let $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a function which maps the 2-dimensional xy-plane into the 2-dimensional $r\theta$-plane.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \vec{f}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}, \text{where } 0 < r < \infty, 0 \le \theta < 2\pi$$

The image of $\vec{f}$ is the entirety of $\mathbb{R}^2$ except the origin. The numbers r and θ are known as **polar coordinates**. The function $\vec{f}$ also has an inverse function $\vec{g}$, which means we can map FROM polar coordinates back to cartesian xy-coordinates (but you have to be careful about inverting the trig functions)

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \vec{g}(x,y) = \begin{bmatrix} \sqrt{x^2+y^2} \\ \arctan(\frac{y}{x}) + k\pi \end{bmatrix}, \text{where } x \ne 0, k \in Z$$

Spherical Coordinates

Let $\vec{s} : \mathbb{R}^3 \to \mathbb{R}^3$ be a function which maps the 3-dimensional xyz-plane into the 3-dimensional $r\phi\theta$-plane.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{s}(r,\phi,\theta) = \begin{bmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{bmatrix}, \text{where } 0 < r < \infty, \quad 0 \le \phi < \pi, \quad 0 \le \theta < \pi$$

Here the image of $\vec{s}(r,\phi,\theta)$ is all of $\mathbb{R}^3$ except the z-axis. The numbers r, ϕ and θ are known as **spherical coordinates**. Like with polar co-ordinates we can map FROM spherical coordinates TO cartesian coordinates by inverting $\vec{s}$, which we will call $\vec{t}$.

$$\begin{bmatrix} r \\ \phi \\ \theta \end{bmatrix} = \vec{t}(x,y,z) = \begin{bmatrix} \sqrt{x^2+y^2+z^2} \\ \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) \end{bmatrix}, \text{where } y \ge 0, x^2+y^2 > 0$$

Notice that when $\phi = 0$ spherical coordinates are identical to polar coordinates.

Cylindrical Coordinates

Cylindrical coordinates are basically the three dimensional version of polar co-ordinates where a 3rd coordinate z is added.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}, \text{where } 0 < r < \infty, \quad -\pi < \theta \le \pi, \quad -\infty \le z < \infty$$

EXAMPLE 1

Let's compute the Jacobian (derivative matrices) of the standard transformations to Cartesian coordinates from polar, spherical and cylindrical coordinates.

Exercise 1

Find the determinant of the Jacobian matrices of the standard polar, spherical and cylindrical coordinate transformations.

CHAPTER 6 REVIEW EXERCISES

Williamson & Trotter, page 309, #6. Find $\frac{\partial}{\partial x} f(K(x,y), K(x+y, x-y), x^2+y^2)$ where $f(x,y,z) = x^2 + y^2 - z^2$ and $K(x,y) = xy$.

Williamson & Trotter, page 311, #42. Find the points on $x^2 + 2xy + 3y^2 = 14$ which are closest to and furthest from the origin.