Multivariable Calculus

Math 212 Fall 2005 Fowler 307 MWF 9:30pm - 10:25am

 http://faculty.oxy.edu/ron/math/212/05/

Class 20: **Wednesday October 26**

SUMMARY Extrema of Multivariable Functions
CURRENT READING Williamson & Trotter, Section 6.4
HOMEWORK Williamson & Trotter, page 292: 2, 3, 7, 9, 12, 20, 21, 26 **Extra Credit page 293: 29, 32, 36**

DEFINITIONS

$f(\vec{x})$ has a **global minimum** at $\vec{x}_0$ if for all $\vec{x}$ in the domain of f , $f(\vec{x}) \geq f(\vec{x}_0)$
$f(\vec{x})$ has a **global maximum** at $\vec{x}_0$ if for all $\vec{x}$ in the domain of f , $f(\vec{x}) \leq f(\vec{x}_0)$
$f(\vec{x})$ has a **local minimum** at $\vec{x}_0$ if there exists a neighborhood N of $\vec{x}_0$, such that $f(\vec{x}) \geq f(\vec{x}_0)$
$f(\vec{x})$ has a **local maximum** at $\vec{x}_0$ if there exists a neighborhood N of $\vec{x}_0$, such that $f(\vec{x}) \leq f(\vec{x}_0)$
A maximum or minimum value of f is called an **extreme value** or **extremum**. The point at which the extremum occurs is known as the **extreme point**.

THEOREM

If a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is defined on a closed, bounded subset S then f assumes its absolute maximum and absolute minimum values on S.

THEOREM

If a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ has a local extreme value at a point $\vec{x}_0$ interior to its domain, then $\vec{\nabla} f(\vec{x}_0) = \vec{0}$ and $\vec{x}_0$ is called a **critical point** of f.

DEFINITION: Quadratic Taylor Polynomial Approximation

A **quadratic Taylor polynomial approximation** of a function $f(x, y)$ near an extreme point (x_0, y_0) where $\vec{\nabla} f(x_0, y_0) = \vec{0}$ can be written as

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2$$
$$+ f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2$$

which can be simplified to

$$f(x,y) \approx f(x_0,y_0) + \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2$$

Since almost every functions behaves like its quadratic Taylor polynomial approximation near its critical points, we can analyze the behavior of quadratic polynomials near their critical points in order to classify general behavior near a critical point.

Exercise 1

Consider $Q(x, y) = ax^2 + bxy + cy^2$. What are its critical points? What is the significance of the value of $D = b^2 - 4ac$?

EXAMPLE 1

Let's show that $Q(x, y) = ax^2 + bxy + cy^2$ can be re-written as

$$Q(x, y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

What can we say about the extreme values of the surface $Q(x, y)$ when $D = b^2 - 4ac > 0$? How about when $D < 0$? How about when $D = 0$?

DEFINITION: saddle point

A critical point $\vec{x}_0$ of $f(\vec{x})$ such that $f(\vec{x}_0)$ is neither a local maximum nor a local minimum value for f is called a **saddle point** for f.

THEOREM: Second Derivative Test

Let $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable. Assume (x_0, y_0) is a critical point (i.e. $\vec{\nabla} f(x_0, y_0) = \vec{0}$).

(i) IF $D > 0$ and $f_{xx}(x_0, y_0) > 0$ or $f_{yy}(x_0, y_0) > 0$, THEN $f(x_0, y_0)$ is a strict local minimum of f.

(ii) IF $D > 0$ and $f_{xx}(x_0, y_0) < 0$ or $f_{yy}(x_0, y_0) < 0$, THEN $f(x_0, y_0)$ is a strict local maximum of f.

(iii) IF $D < 0$, THEN f possesses a **saddle point** at (x_0, y_0).

(iv) IF $D = 0$ THEN anything can happen: f could have a local minimum, a local maximum, a saddle point or none of these at (x_0, y_0).

Exercise 2

Find and classify all the critical points of $f(x, y) = x^3 - 3x + y^3 - 3y$.

EXAMPLE 2

Find the extrema of $f(x, y) = x^2 + y^2$ on the set of points (x, y) in the set S of points which lie inside and on the ellipse $x^2 + 2y^2 = 1$.

Method of Lagrange Multipliers

Suppose a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, restricted to a set S and possesses a local extreme point at $\vec{x}_0$ on S. Suppose that near $\vec{x}_0$, the set S is a smooth level set of a function $G : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ and coordinate functions G_1, G_2, …, G_m. THEN there exist constants λ_1, λ_2, λ_3,…, λ_m such that $\vec{x}_0$ is a critical point of the real valued function $f + \lambda_1 G_1 + \lambda_2 G_2 + \ldots + \lambda_m G_m$ such that

$$\vec{\nabla} f(\vec{x}_0) + \lambda_1 \vec{\nabla} G_1(\vec{x}_0) + \lambda_2 \vec{\nabla} G_2(\vec{x}_0) + \ldots + \lambda_m \vec{\nabla} G_m(\vec{x}_0) = \vec{0}$$

$\boxed{\text{EXAMPLE 2}}$

Let's do EXAMPLE 1 again, this time using Lagrange Multipliers.