Multivariable Calculus

Math 212 Fall 2005 Fowler 307 MWF 9:30pm - 10:25am
 http://faculty.oxy.edu/ron/math/212/05/

Class 18: **Wednesday October 19**

SUMMARY The Multivariable Chain Rule
CURRENT READING Williamson & Trotter, Section 6.2
HOMEWORK Williamson & Trotter, page 269: 2, 3, 4, 5, 7, 11, 12, 13 **Extra Credit page 271: 26**

THEOREM: The Chain Rule

Let U be an open subset of $\mathbb{R}^n$ and V be an open subset of $\mathbb{R}^m$. Let $\vec{f} : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{g} : V \subset \mathbb{R}^m \to \mathbb{R}^p$ be given functions such that $\vec{f}$ maps U into V and $\vec{g} \circ \vec{f}$ is defined. Suppose $\vec{g}$ is differentiable at $\vec{x}_0$ and $\vec{f}$ is differentiable at $\vec{y}_0 = \vec{g}(\vec{x}_0)$ Then $\vec{g} \circ \vec{f}$ is differentiable at $\vec{x}_0$ and

$$(\vec{g} \circ \vec{f})' = \vec{g}'(\vec{y}_0)\vec{f}'(\vec{x}_0)$$

NOTE: the two objects on the right hand side of the above equation are **derivative matrices** (jacobians).
The best way to understand the Chain Rule is with lots of examples...

EXAMPLE 1

Let's consider an example where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. $w = f(x) = e^{-x^2}$ and $x = g(t) = \sin(3t + \pi)$. What is $\dfrac{dw}{dt}$?

EXAMPLE 2

Consider $f(x, y, z) = zx + y^2 + x^3 + \ln(x + y)$ and $x(s, w) = we^{-2s}$, $s(t) = 7t - 9$, $y(t) = \sin(t)$. Can we find $\dfrac{\partial f}{\partial t}$?

$\boxed{\textbf{Exercise 1}}$

Consider $\vec{f}(x,y) = (x^2+y^2, x^2-y^2)$ and $\vec{g}(u,v) = (uv, u+v)$.

(a) Find $\vec{g} \circ \vec{f}$. Are the input variables (x,y) or (u,v)?

(b) Find the Jacobian of $\vec{g} \circ \vec{f}$

(c) Compute the Jacobian of f and the Jacobian of g

(d) Apply the Chain Rule to show that $(\vec{g} \circ \vec{f})' = g'(\vec{f})f'$

$\boxed{\text{EXAMPLE 3}}$

Consider $x = u^2 + v^3, y = e^{uv}, z = u - v$ and $u = t+1, v = e^t$. Find $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ and $\dfrac{dz}{dt}$ at $t = 0$

$\boxed{\textbf{Exercise 2}}$

Williamson & Trotter, page 271 #20-#25. Consider the following $\vec{f}$ and $\vec{g}$ and decide whether $\vec{f} \circ \vec{g}$ or $\vec{g} \circ \vec{f}$ or neither one, can possibly be defined:

#20 $f : \mathbb{R}^2 \to \mathbb{R}^2, g : \mathbb{R}^2 \to \mathbb{R}^3$
#21 $f : \mathbb{R}^3 \to \mathbb{R}^2, g : \mathbb{R}^2 \to \mathbb{R}$
#22 $f : \mathbb{R} \to \mathbb{R}^2, g : \mathbb{R} \to \mathbb{R}^2$
#23 $f : \mathbb{R}^3 \to \mathbb{R}^2, g : \mathbb{R}^3 \to \mathbb{R}^3$
#24 $f : \mathbb{R} \to \mathbb{R}^2, g : \mathbb{R}^3 \to \mathbb{R}^2$
#25 $f : \mathbb{R} \to \mathbb{R}^3, g : \mathbb{R}^3 \to \mathbb{R}^3$

Visualizing The Chain Rule

(Adapted from *Multivariable Calculus* by McCallum, Hughes-Hallet, et al, 2005).

Suppose z depends on x and y, both of which depend on t, we can draw a diagram depicting these relationships below:

There are lines between z and both x and y and then since x and y are both functions of t there is a line from x to t and from y to t. Along each of these lines write down the appropriate derivative. By following all the possible paths from z to t and adding them up one obtains the following expression for $\frac{dz}{dt}$ where

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Exercise 3

Draw the diagram for $z = f(x, y)$ where $x = g(u, v)$ and $y = h(u, v)$.

Use the Chain Rule (assisted by the diagram) to write down the Chain Rule for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

General Chain Rule Diagram

To find the rate of change of one variable with respect to another in a "chain" of differentiable functions which are compositions of each other:

1. Draw a diagram expression the relationship between the variables and lable each link in the diagram with the derivative relating the variables at each end.
2. For each path between two variables, multiple together the derivatives fro each step along the path.
3. Add the contributions from each path.

$\boxed{\text{EXAMPLE 4}}$

Math 224 Spring 2003, Exam 1, Question 10. Draw a tree of variables for which this is the correct chain rule:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{dy}{du}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$\boxed{\textbf{Exercise 4}}$

Repeat EXAMPLE 3, this time using the Visual Chain Rule to assist you with your computations. Recall, $x = u^2 + v^3, y = e^{uv}, z = u - v$ and $u = t + 1, v = e^t$. You want to find $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ at $t = 0$