Multivariable Calculus

Class 17: **Monday October 17**

SUMMARY Gradient Fields and Vector Fields
CURRENT READING Williamson & Trotter, Section 6.1 and 6.2
HOMEWORK Williamson & Trotter, page 257: 5, 6, 8, 9, 17, 20, 26, 27 page 261: 4, 8,
Extra Credit page 259: 38

THEOREM

Let $\vec{f} : \mathbb{R}^n \to \mathbb{R}$ be differentiable in an open set D in $\mathbb{R}^n$. Then at each point $\vec{x}$ in D for which $\vec{\nabla} f(\vec{x})$ points in the direction of maximum increase for f. The number $|\vec{\nabla} f(\vec{x})|$ is the maximum rate of increase of f at the point $\vec{x}$.

EXAMPLE

Consider $f(x, y) = x^2 + y^2$. The gradient $\vec{\nabla} f(\vec{x}) = 2x\hat{i} + 2y\hat{j}$. This means that at every point (x, y) we can draw a vector which corresponds to the gradient. This is known a **vector field** and can be sketched in the space below. Vector fields which are obtained by using the gradient operator are called **gradient fields**.

Exercise

Consider $\vec{F}(\vec{x}) = (-y, x)$ for $x^2 + y^2 \leq 4$. Sketch this vector field below. Is it a gradient field?

A great web resource for visualizing Vector Fields is the **Java Vector Field Analyzer II**, available from the Math 212 Course Website resources page.

Normal Vector
Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with $n \geq 2$ with f continuously differentiable at $\vec{x}_0$. Let S be a level set of f containing $\vec{x}_0$. If $\vec{\nabla} f(\vec{x}_0) \neq 0$, then $\vec{\nabla} f(\vec{x}_0)$ is a normal vector to S at $\vec{x}_0$, and all points in a tangent (line) plane to S at $\vec{x}_0$ satisfy the equation

$$\vec{\nabla} f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

EXAMPLE
Let's find the equation of the tangent line to the $k = 25$ level set of $f(x, y) = x^2 + y^2$ at the point $(3, 4)$.

Do you notice that the slope of the tangent line is orthogonal to the direction of the gradient vector, at the point $(3, 4)$? How can you show this?

These ideas are combined to produce the following theorem. This theorem is the basis for a very important technique for solving nonlinear problems called the **Method of Steepest Descent**.

THEOREM
The direction of maximum increase of a differentiable function f at $\vec{x}_0$ is perpendicular to the level set of f containing $\vec{x}_0$, assuming $\vec{\nabla} f(\vec{x}_0) \neq 0$.

Flow Lines
Consider a continuously differentiable vector field $\vec{F}(\vec{x})$ is defined on an open subset S of $\mathbb{R}^n$. The parametrized curve $\vec{x} = \vec{g}(t)$ is called a **flow line** of $\vec{F}(\vec{x})$ if the velocity vector $\frac{d\vec{x}}{dt}$ at a point $\vec{x}$ in S coincides with the vector $\vec{F}(\vec{x})$, in other words, if $\frac{d\vec{x}}{dt} = \vec{F}(\vec{g}(t))$

Exercise
Williamson & Trotter, page 261, #7. **(a)** Verify that the curve parametrized by $\vec{x}(t) = (a\cos t + b\sin t, b\cos t - a\sin t)$ is a flow line of the vector field $\vec{F}(\vec{x}) = (y, -x)$. **(b)** Show that these flow lines are circles.