Class 14: **Friday October 7**

SUMMARY The Jacobian Matrix
CURRENT READING Williamson & Trotter, Section (Section 5.4)
HOMEWORK Williamson & Trotter, page 236: # 1, 4, 9; page 243: 3, 4, 8, 17, 18, 26
Extra Credit page 244: 27, 28

Definition: jacobian

The **derivative matrix** (usually called the **jacobian**) of a vector function $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ is the matrix consisting of the n partial derivatives of each of the m co-ordinate functions arranged so that the rows of the matrix are exactly gradient vectors of each coordinate function. The Jacobian has mn entries where $J_{i,j} = \frac{\partial f_i}{\partial x_j}$. In other words,

$$J(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

EXAMPLE 1

Williamson & Trotter, page 243, #1. Find the "derivative matrix" $\vec{f}'$ at a general point of the domain of the function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ where $f(x, y) = (xy, x + y)$

Exercise

Williamson & Trotter, page 243, #2. Find the "derivative matrix" $\vec{f}'$ at a general point of the domain of the function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ where $f(r, \theta) = (r\cos(\theta), r\sin(\theta))$

First Degree Taylor Approximations For Different Types of Functions

We know that for a scalar function of a scalar variable ($f : \mathbb{R} \to \mathbb{R}$) one can approximate the function $f(x)$ near x_0 with the equation of the tangent line:

$$T(x) = f(x_0) + f'(x_0)(x - x_0)$$

For a scalar function of a vector variable ($f : \mathbb{R}^m \to \mathbb{R}$) the tangent approximation becomes

$$T(\vec{x}) = f(\vec{x}_0) + \vec{\nabla} f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$

For a vector function of a scalar variable ($f : \mathbb{R} \to \mathbb{R}^n$) the tangent approximation is

$$\vec{T}(t) = \vec{f}(t_0) + \frac{d\vec{f}}{dt}(t_0)(t - t_0)$$

For a vector function of a vector variable ($f : \mathbb{R}^m \to \mathbb{R}^n$) the tangent approximation is

$$\vec{T}(\vec{x}) = \vec{f}(\vec{x}_0) + J(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

GROUPWORK

Write down YOUR OWN FOUR DIFFERENT examples of the four different types of function and take their "effective derivative" in each case. Compare your examples with your nearest neighbors.

EXAMPLE 2

Williamson & Trotter, page 243, #25. Consider the function from Example 1 (i.e. $\vec{f}(x, y) = (xy, x + y)$ with $\vec{x}_0 = (1, 0)$, $\vec{y}_1 = (0.1, 0)$, $\vec{y}_2 = (0, 0.1)$ and $\vec{y}_3 = (0.1, 0.1)$

(a) Compute $\vec{f}(\vec{x}_0 + \vec{y}_i)$ for $i = 1, 2, 3$

(b) Find the tangent approximation to $\vec{f}(\vec{x}_0 + \vec{y})$ for an arbitrary vector $\vec{y}$.

(c) Use the tangent approximation from (b) to approximate the vectors $\vec{f}(\vec{x}_0 + \vec{y}_i)$ for $i = 1, 2, 3$