Multivariable Calculus

Class 11: **Friday September 30**

SUMMARY Application of Partial Derivatives: Vector Partial Derivatives and Quadric Surfaces
CURRENT READING Williamson & Trotter, Section (Section 4.3)
HOMEWORK Williamson & Trotter, page 203: 3, 9, 12, 22, 25, 31, 34, 37; **Extra Credit page 204: # 38, 39**

DEFINITION

A vector function $f : \mathbb{R}^n \to \mathbb{R}^m$ of a vector variable, $\vec{f}(\vec{x})$, has a **vector partial derivative** given by

$$\frac{\partial \vec{f}}{\partial x_i} = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_m}{\partial x_i} \right)$$

EXAMPLE 1

Williamson & Trotter, page 210, #1. Find formulas for the vector partial derivative of the function $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3$ where $\vec{f}(x, y) = \begin{bmatrix} x + y \\ x - y \\ x^2 + y^2 \end{bmatrix}$

Graphical Interpretation of Vector Partial Derivative

If you think about a vector function of a vector variable only varying with respect to one variable while the rest of the variables are held constant then you could reframe this as a vector function of that one scalar variable, and its image would trace out a **coordinate curve** in $\mathbb{R}^m$. The vector partial derivative $\dfrac{\partial \vec{f}}{\partial x_i}$ then is exactly the same thing as the tangent vector to this coordinate curve.

Parametrized Surfaces

A curved surface $z = f(x, y)$in $\mathbb{R}^3$ can be represented in a parametrized way as $\vec{x}(u, v) = \vec{f}(u, v)$ where $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3$. This is the multi-dimensional equivalent to the way a curve in $\mathbb{R}^2$ can be represented parametrically by $\vec{x}(t) = \vec{f}(t)$ where $\vec{f} : \mathbb{R} \to \mathbb{R}^2$.

Quadric Surfaces

Quadric Surfaces are level sets in $\mathbb{R}^3$ of second degree polynomials in three variables. In other words they have the form $Ax^2 + By^2 + Cz^2 + D = 0$. There are six different distinct types of quadric surfaces, all of which are displayed on page 196 of Williamson & Trotter. There's a great interactive web resource called the "Interactive Gallery of Quadric Surfaces." The curves are:

hyperboloid of one sheet $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z^2 = k > 0$

hyperboloid of two sheets $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z^2 = k < 0$

elliptic cone $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z^2 = 0$

ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

elliptic paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z = 0$

hyperbolic paraboloid $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - z = 0$

EXAMPLE 1

Show that an **elliptic cone** can be represented parametrically by $\vec{x}(u,v) = \begin{bmatrix} au\cos v \\ bu\sin v \\ u \end{bmatrix}$

When a surface is represented parametrically, the tangent plane can be represented as

$$\vec{x} = u\frac{\partial \vec{x}}{\partial u}(u_0, v_0) + v\frac{\partial \vec{x}}{\partial v}(u_0, v_0) + \vec{x}(u_0, v_0)$$

Exercise

Show that the parametrized surface $\vec{x} = \begin{bmatrix} 2\cos u \sin v \\ 3\sin u \sin v \\ 4\cos v \end{bmatrix}$ represents the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1$ and find the parametrized equation of the tangent plane at $\vec{x}(0,0)$.