Multivariable Calculus

Math 212 Fall 2005 — Fowler 307 MWF 9:30pm - 10:25am

 — http://faculty.oxy.edu/ron/math/212/05/

Class 4: **Monday September 12**

SUMMARY Euclidean Geometry and The Vector Cross Product
CURRENT READING Williamson & Trotter, Section 1.5 and Section 1.6
HOMEWORK Williamson & Trotter, §1.6 # 1, 2, 3, 4, 8, 9, 11, 22

General Equation of a Plane in Euclidean Space

The main way we often think of planes in euclidean space (i.e. the space we are used to living in where lines are perfectly "straight" and go on forever) is to define a plane in $\mathbb{R}^3$ which contains the point $\vec{x}_0$ as that 2-D object which is exactly perpendicular to a particular vector $\vec{p}$:

$$\vec{p} \cdot (\vec{x} - \vec{x}_0) = 0$$

Note that in $\mathbb{R}^2$ you can define the equation of a line this way, also.
Another way to write this is to think of the vector emanating from the plane normal (i.e. perpendicularly) to the plane as a unit vector $\hat{n} = \vec{p}/|\vec{p}|$ so that

$$\hat{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

or

$$\hat{n} \cdot \vec{x} - c = 0$$

Note, that this form of a plane means that when you see the other standard form $Ax + By + Cz = D$ then you know the vector $\vec{p} = A\hat{i} + B\hat{j} + C\hat{k}$ is perpendicular to the plane and that $D = Ax_0 + By_0 + Cz_0$.

In fact, two planes are considered parallel to each other if their normal vectors $\hat{n}_1$ and $\hat{n}_2$ are parallel.

EXERCISE 1

Consider $2x + 3y - 4z = 6$ and $4x + 6y - 8z = 9$. Are these two objects lines or planes? Are they parallel to each other? Write down an example of a vector which is perpendicular to each one of these objects.

Distance from a Point to a Plane

The distance δ from a point in space $\vec{x}_1$ to a plane (or line) $\hat{n} \cdot (\vec{x} - \vec{x_0}) = 0$ or $\hat{n} \cdot \vec{x} - c = 0$ is given by

$$\delta = \hat{n} \cdot (\vec{x}_1 - \vec{x}_0) = \hat{n} \cdot \vec{x}_1 - c = 0$$

$\boxed{\textbf{EXERCISE 2}}$

What is the distance between the point **P** $(1, 1, 1)$ and the plane $x + y - 2z = -1$? Which point is closer to the plane, **P** or the origin?

The Vector Cross Product

$\boxed{\text{DEFINITION}}$

The **cross product** of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is defined to be the vector

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

Luckily, there's an easy way to remember this calculation as the determinant of a 3x3 matrix

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

$\boxed{\text{EXAMPLE}}$

Find the cross product $\vec{u} \times \vec{v}$ of $\vec{u} = (1, -3, 2)$ and $\vec{v} = (2, 4, -5)$.

Take the dot product of your answer with both $\vec{u}$ and $\vec{v}$. What do you notice?

Properties of The Cross Product

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$$

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$\hat{i} \times \hat{j} = k; \hat{j} \times \hat{k} = \hat{i}; \hat{k} \times \hat{j} = \hat{i}$$

Additivity: $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ and $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
Homogeneity: $r(\vec{u} \times \vec{v} = (r\vec{u}) \times \vec{v} = \vec{u} \times (r\vec{v})$
Area of a Parallelogram $= |\vec{u} \times \vec{v}|$
Volume of a Parallelpiped $= \vec{u} \cdot (\vec{v} \times \vec{w})$